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On a class of semilinear fractional elliptic equations involving outside Dirac data

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ABSTRACT

The purpose of this article is to give a complete study of the weak solutions of the fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 \quad \text{in } B_1(e_N), \\ u &= \delta_0 \quad \text{in } \mathbb{R}^N \setminus B_1(e_N), \end{aligned} \tag{0.1}$$

where $p \geq 0$, $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ denotes the fractional Laplacian operator in the principle value sense, $B_1(e_N)$ is the unit ball centered at $e_N = (0,\cdots,0,1)$ in $\mathbb{R}^N$ with $N \geq 2$ and δ_0 is the Dirac mass concentrated at the origin. We prove that problem (Eq. (0.1)) admits a unique weak solution when $p > 1 + \frac{2\alpha}{N}$. Moreover, if in addition $p \geq \frac{N+2}{N-2}$, the weak solution vanishes as $\alpha \to 1^-$. We also show that problem (Eq. (0.1)) does not have any weak solution when $p \in [0, 1 + \frac{2\alpha}{N}]$. These results are very surprising since they are in total contradiction with the classical setting, i.e.

$$\begin{aligned} -\Delta u + u^p &= 0 \quad \text{in } B_1(e_N), \\ u &= \delta_0 \quad \text{in } \mathbb{R}^N \setminus B_1(e_N), \end{aligned}$$

for which it has been proved that there are no solutions for $p \geq \frac{N+1}{N-1}$.

1. Introduction

Fractional PDEs have gained tremendous interest, not only from mathematicians but also from physicists and engineering, during the last years. This is essentially due to their widespread domains of applications. In

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fact the fractional Laplacian arises in many areas including medicine [12], bio-engineering [18–21], relativistic physics [1,16,17], Modeling populations [27], flood flow, material viscoelastic theory, biology and earthquakes. It is also particularly relevant to study some situations, in which the fractional Laplacian is involved in PDEs, featuring irregular data such that those phenomena describing source terms which are concentrated at points. In our context, the source is placed outside the unit ball $B_1(e_N)$. This generates long-term interactions and short-term interactions, described by the nonlocal operator $(-\Delta)^\alpha$ and the nonlinear absorption u^p respectively. $(-\Delta)^\alpha$ has also a probabilistic interpretation, related to the above one. It is the α-stable subordinated infinitesimal killed Brownian motion.

Let $B_1(e_N)$ be the unit ball in $\mathbb{R}^N(N \geq 2)$ with center $e_N = (0, \dots, 0, 1)$ and δ_0 be the Dirac mass concentrated at the origin. Our main objective in this article is to investigate the existence, nonexistence and uniqueness of positive weak solutions of the semilinear fractional equation

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 \quad \text{in } B_1(e_N), \\ u &= \delta_0 \quad \text{in } \mathbb{R}^N \setminus B_1(e_N), \end{aligned} \tag{1.1}$$

where $p \geq 0$ and the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$ is defined by

$$(-\Delta)^\alpha u(x) = c_{N,\alpha} \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

where

$$c_{N,\alpha} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(z_1)}{|z|^{N+2\alpha}} dz \right)^{-1} \tag{1.2}$$

with $z = (z_1, \dots, z_N) \in \mathbb{R}^N$ and

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} dz.$$

In 1991, a fundamental contribution to semilinear elliptic equations involving measures as boundary data is due to Gmira and Véron [15], where they studied the existence and uniqueness of weak solutions for

$$\begin{aligned} -\Delta u + h(u) &= 0 \quad \text{in } \Omega, \\ u &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where Ω is a bounded C^2 domain and μ is a bounded Radon measure defined in $\partial\Omega$. A function u is said to be a weak solution of (1.3) *if* $u \in L^1(\Omega)$, $h(u) \in L^1(\Omega, \rho dx)$ *and*

$$\int_\Omega [u(-\Delta)\xi + h(u)\xi] dx = \int_{\partial\Omega} \frac{\partial \xi(x)}{\partial \vec{n}_x} d\mu(x), \quad \forall \xi \in C_0^{1.1}(\Omega), \tag{1.4}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ *and* $\vec{n}_x$ *denotes the unit inward normal vector at a point* x. Gmira and Véron proved that the problem (1.3) admits a unique weak solution when h is a continuous and nondecreasing function satisfying

$$\int_1^\infty [h(s) - h(-s)] s^{-1-\frac{N+1}{N-1}} ds < +\infty. \tag{1.5}$$

The weak solution of (1.3) is approached by the classical solutions of (1.3) when μ is replaced by a sequence of regular functions $\{\mu_n\}$, which converge to μ in the distribution sense. Furthermore, they showed that there is no weak solution of (1.3) when $\mu = \delta_{x_0}$ with $x_0 \in \partial\Omega$ and $h(s) = |s|^{p-1}s$ with $p \geq \frac{N+1}{N-1}$. Later on, this subject has been vastly expanded in recent works, see the papers of Marcus and Véron [22–25], Bidaut-Véron and Vivier [2] and references therein.