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Optimal distributed controls of a class of nonlinear dispersive equations with cubic nonlinearity☆

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ABSTRACT

This paper is devoted to the optimal distributed control problem governed by a class of nonlinear dispersive equations with cubic nonlinearity, which contains the famous Novikov equation as special case. We first investigate the existence and uniqueness of the weak solution for the controlled system, and then find an optimal solution for the controlled system with the generalized cost functional. Moreover, by means of the method suggested by A.Ya. Dubovitskii and A.A. Milyutin, we establish the first order necessary optimality condition of optimal control for the controlled system in the fixed final horizon case.

1. Introduction

In present work, we mainly focus on the optimal distributed control problem governed by following a class of nonlinear dispersive equations with cubic nonlinearity,

$$u_t(x,t) - u_{txx}(x,t) + (b+1)u^2(x,t)u_x(x,t) = u^2(x,t)u_{xxx}(x,t) + bu(x,t)u_x(x,t)u_{xx}(x,t), \tag{1.1}$$

where $(x,t) \in (0,1) \times (0,T)$ and $b \in \mathbb{R}$ is a constant [41]. The unknown function $u(x,t)$ denotes the velocity field in the spatial x direction at time t.

For $b = 3$, Eq. (1.1) becomes the well-known Novikov equation

$$u_t(x,t) - u_{txx}(x,t) + 4u^2(x,t)u_x(x,t) - u^2(x,t)u_{xxx}(x,t) - 3u(x,t)u_x(x,t)u_{xx}(x,t) = 0, \tag{1.2}$$

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which was discovered very recently by V. Novikov in a symmetry classification of nonlocal partial differential equations with quadratic or cubic nonlinearity [44]. After the Novikov equation was derived, many papers were devoted to its study from the mathematical point of view. For example, in [33], Hone and Wang found a matrix Lax pair for the Novikov equation, which was shown to be concerned with a negative flow in the Sawada–Kotera hierarchy. In [32], Hone et al. found non-smooth explicit soliton solutions with multiple peaks for (1.2) by utilizing the scattering theory. This property is also valid for some special b-equations (see (1.3)), for example the Degasperis–Procesi (DP) equation and the Camassa–Holm (CH) equation, see for instance [3,15–17]. A peak solution is proved to be a global weak solution for a modified Novikov equation by Mi and Mu [41]. In [34], due to the special construction of Novikov equation, sufficient conditions imposed on the initial data are established to ensure the formation of singularities in finite time. In [28], Himonas and Holliman considered the initial value problem of (1.2) in the Sobolev space $H^s(\mathbb{R})$ on both the circle $\mathbb{T}$ and the line $\mathbb{R}$ for $s > 3/2$. In [27], Grayshan investigated the non-periodic as well as the periodic Cauchy problems for Eq. (1.2) in the Sobolev space with exponent less than 3/2. In [43], Ni and Zhou proved the local well-posedness for the Novikov equation in the Besov spaces $B^s_{2,r}$ with the critical index $s = 3/2$, and they studied the well-posedness in H^s by applying Kato's semigroup theory for $s > 3/2$. Moreover, they also obtained two results concerning the persistence properties of the strong solution. When the Sobolev index $s \geq 3$, the orbit invariants were adopted in [54] to investigate the existence of a periodic global strong solution with a sign condition. The Cauchy problem of the Novikov equation with analytic initial data was studied in [54]. In [36], if the initial data satisfies a sign condition, Lai et al. showed that the Novikov equation (1.2) admits a unique global weak solution in the Sobolev space $H^s(\mathbb{R})$ with $1 \leq s \leq 3/2$. Without the sign condition imposed on the initial data, Lai [35] studied the existence of global weak solutions to the Cauchy problem for the Novikov equation in the space $C([0,\infty];\mathbb{R}) \cap L^\infty([0,\infty);H^1(\mathbb{R}))$. Under certain assumptions, Yan et al. [56] showed that Eq. (1.2) is locally well-posed in the Besov space by utilizing the Littlewood–Paley decomposition approach. For the other works related to the Novikov equation, we refer the readers to [4,9,10,19,38,39,55] and the references therein.

By comparing Eq. (1.1) with the b-equation in the form of

$$u_t(x,t) - u_{txx}(x,t) + (b+1)u(x,t)u_x(x,t) = bu(x,t)u_{xx}(x,t) + u(x,t)u_{xxx}(x,t), \tag{1.3}$$

where $b \in \mathbb{R}$ is a real constant, it is clear that Eq. (1.1) contains cubic nonlinear terms (rather than quadratic of b-equation). By an appropriate Kodama transformation, the b-equation can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$, and the asymptotic ordering is violated and the corresponding Kodama transformation is singular for $b = -1$, see for example [21,22].

Especially, if we take $b = 2$ in Eq. (1.3), then it becomes the famous CH equation (if $b = 3$, it becomes the (DP) equation [16])

$$u_t(x,t) - u_{txx}(x,t) + 3u(x,t)u_x(x,t) - 2u_x(x,t)u_{xx}(x,t) - u(x,t)u_{xxx}(x,t) = 0, \tag{1.4}$$

where the unknown function $u(x,t)$ describes the free surface of the water above a flat bottom. It is well known that the CH equation is a completely integrable equation, which has an infinite conservation laws as well as a bi-Hamilton structure [3,5,29], and it also has global dissipative and conservative solutions [2,30,31]. During past several years, some papers are available to investigate the dynamic properties of Eq. (1.4). For example, it is shown in [6,13,17] that the scattering approach as well as the inverse spectral are very useful methods to research the dynamic properties of the CH equation. The CH equation has travelling water waves of largest amplitude (the Stokes waves of greatest height); for the CH model these solutions take the form of $ce^{-|x-ct|}$, see for instance [4,9,10,19]. It is also worthwhile noting that the CH equation leads to the geodesic flow of a certain invariant metric on the Bott–Virasoro group [15,42], which implies