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Dirac-harmonic equations for differential forms

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ABSTRACT

In this paper, we introduce the Dirac-harmonic equation for differential forms and prove some basic norm inequalities, including the Caccioppoli-type inequality, Poincaré-type inequality and the weak reverse Hölder inequality for the solutions of this kind of differential equations.

1. Introduction

The purpose of this paper is to introduce the Dirac-harmonic equation $d^*A(x, D\omega) = 0$ for differential forms and initiate the study of this new type of differential equations, where the Hodge–Dirac operator D is defined by $D = d + d^*$, d is the exterior differential operator, d^* is the Hodge codifferential that is formal adjoint operator of d, and A is an operator satisfying certain conditions. Specifically, we establish the Poincaré–Sobolev inequalities, L^P-imbedding inequalities, Caccioppoli-type inequalities and the weak reverse Hölder inequality for differential forms satisfying the Dirac-harmonic equation. These basic inequalities will form the basis for the study of the L^p-theory of the new introduced Dirac-harmonic equation for differential forms. As extensions of functions, differential forms have been well studied and used in recent years, see [1,6,4,5,22,23].

A Dirac operator is a differential operator that is a formal square root, or half-iterate, of a second-order operator such as a Laplacian. It was introduced initially by Paul Dirac in studying quantum mechanics

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theory. Later, Atiyah and Singer rediscovered the Dirac operator in studying index theorem. The work was so significant that the operator later played as one of the central roles in modern mathematics and some other disciplines, in particular, as a unifying tool bringing together analysis and geometry under the umbrella of what is now called global analysis [7].

As the consequence of studies, different versions of Dirac operators were introduced in different fields of mathematics and physics. For example, the Dirac operator arising in Clifford analysis is of the form $D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$, where $\{e_j : j = 1, \ldots, n\}$ is an orthonormal basis for the n-dimensional euclidean space, and $\mathbb{R}^n$ is considered to be embedded in a Clifford algebra. There has been much investigation related this form of the operator with either p-harmonic equations or A-harmonic equations, see [19,18,12,16].

We keep using the traditional notations appearing in [1] throughout this paper. Particularly, we use Ω to denote a domain and B to denote a ball in $\mathbb{R}^n$, $n \geq 2$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$ and $\wedge = \wedge(\mathbb{R}^n) = \oplus_{l=0}^{n} \wedge^l((\mathbb{R}^n))$ be a graded algebra with respect to the exterior products. Assume that $D'(\Omega, \wedge^l)$ is the space of all differential l-forms in Ω and $L^p(\Omega, \wedge^l)$ is the space of all l-forms $u(x) = \sum_I u_I(x) dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We always use $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$, to denote the exterior derivative. The Hodge star operator $\star : \wedge^k \to \wedge^{n-k}$ is defined as follows. If $\omega = \omega_{i_1 i_2 \cdots i_k}(x_1, x_2, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = \omega_I dx_I$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star\omega = \star(\omega_{i_1 i_2 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^{\sum(I)} \omega_I dx_J$, where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = \frac{k(k+1)}{2} + \sum_{j=1}^{k} i_j$. The Hodge codifferential operator $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is defined by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$.

The theory of differential equations has been very well developed during last several decades. Particularly, there has been an increasing interest in different types of differential equations for differential forms, see [2,3,20,13,8,15,14]. Among these types of equations, the traditional A-harmonic equation for differential forms

$$d^* A(x, d\omega) = 0 \tag{1.1}$$

has received much investigation in recent years. However, in many situations, we have to deal with terms $d\omega$ and $d^*\omega$, such as in the case of Hodge decomposition of a differential form. Hence, we are motivated to introduce the following Dirac-harmonic equation for differential forms

$$d^\star A(x, D\omega) = 0, \tag{1.2}$$

where $D = d + d^*$ is the Dirac operator and $A : \Omega \times \wedge(\mathbb{R}^n) \to \wedge(\mathbb{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.2). Let $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1}) = \cap W^1_p(\Omega', \wedge^{l-1})$, where the intersection is for all Ω' compactly contained in Ω. A solution to (1.2) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, D\omega), D\varphi \rangle = 0 \tag{1.4}$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

A solution ω of the Dirac-harmonic equation (1.2) is called a nontrivial solution if $D\omega \neq 0$; otherwise, ω is called a trivial solution of (1.2). Similarly, a solution ω of the A-harmonic equation (1.1) is called a nontrivial solution if $d\omega \neq 0$; otherwise, ω is called a trivial solution of (1.1). It should be noticed that the Dirac-harmonic equation can be considered as an extension of the traditional A-harmonic equations with operator d being replaced by the Dirac operator $D = d + d^*$. It is also easy to see that if ω is a function (0-form), both the traditional A-harmonic equation $d^* A(x, d\omega) = 0$ and the Dirac-harmonic equation $d^* A(x, D\omega) = 0$ reduce to the usual A-harmonic equation

$$\mathrm{div} A(x, \nabla\omega) = 0 \tag{1.5}$$