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On the regularity criterion for the Navier–Stokes equations involving the diagonal entry of the velocity gradient

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ABSTRACT

In this paper, we provide a regularity criterion for the Navier–Stokes equations involving the diagonal entry of the velocity gradient, which says that if

$$\partial_3 u_3 \in L^\infty(0,T;L^2(\mathbb{R}^3)),$$

then the weak solution is smooth on $(0,T)$. This is an important case in the sense that the strong solution is in this class. Moreover, we verify the limiting case of one regularity criterion in Cao and Titi (2011).

1. Introduction

This paper concerns itself with the following three-dimensional (3D) incompressible Navier–Stokes equations

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla\pi &= \boldsymbol{0}, \\ \operatorname{div}\boldsymbol{u} &= 0, \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, \end{cases} \tag{1}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the fluid velocity field, π is a scalar pressure, and $\boldsymbol{u}_0$ is the prescribed initial data.

Since the pioneering work of Leray [16] and Hopf [10] on the global existence of a weak solution $\boldsymbol{u}$ (now being called the Leray–Hopf weak solution, see Definition 1), there have been a lot of studies devoted to studying the regularity of such solutions. Prodi [21], Serrin [22] and Ladyzhenskaya [14] first showed that if

$$\boldsymbol{u}\in L^p(0,T;L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p}+\frac{3}{q}=1,\ 3\le q\le\infty, \tag{2}$$

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then $\boldsymbol{u}$ is smooth on $(0, T)$. Notice that the liming case was covered only in 2003 by Eskauriaza–Serëgin–Sverák [8]. Then H. Beirão da Veiga [1] showed the regularity criterion involving the velocity gradient

$$\nabla \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = 2, \ \frac{3}{2} \le q \le \infty. \tag{3}$$

Note that the limiting case $\nabla \boldsymbol{u} \in L^\infty(0, T; L^{\frac{3}{2}}(\mathbb{R}^3))$ follows from (2) and the Sobolev embedding theorem.

Due to the divergence-free condition, an interesting problem is whether the regularity of the solution can be guaranteed by partial components of the velocity or its gradient. Neustupa–Novotny–Penel [17] and Zhou [29] first proved the following regularity condition

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \frac{1}{2}, \ 6 < q \le \infty. \tag{4}$$

This was improved by Kukavica–Ziane [12] to be

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \frac{5}{8}, \ \frac{24}{5} < q \le \infty, \tag{5}$$

and by Cao–Titi [4] to be

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \frac{2}{3} + \frac{1}{3q}, \ \frac{7}{2} < q \le \infty. \tag{6}$$

Thanks to Zhou–Pokorný [31], we have the finest up-to-date result

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \frac{3}{4} + \frac{1}{2q}, \ \frac{10}{3} < q \le \infty, \tag{7}$$

with the liming case $u_3 \in L^\infty(0, T : L^{\frac{10}{3}}(\mathbb{R}^3))$ treated in [11].

As far as the regularity criteria involving the velocity gradient, Pokorný [20] and Zhou [28] proved that the condition

$$\nabla u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \ 2 \le q \le \infty \tag{8}$$

implies smoothness. For further progresses on this topic, the interested readers are referred to [6,24,23,26,27,30]. Another approach is to show the regularity of the solution under the assumptions on $\partial_3 \boldsymbol{u}$, see [2,13,18,19].

We may also consider the regularity condition involving only one entry of the velocity gradient. In [31], it was proved that if

$$\partial_3 u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} < \frac{4}{5}, \ \frac{15}{4} < q \le \infty, \tag{9}$$

then the solution is regular. By introducing a new method of proof, (9) was improved in [11] to be

$$\partial_3 u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} \le \frac{4}{5}, \ \frac{15}{4} \le q \le \infty. \tag{10}$$

Then Cao–Titi [5] established two general criteria which read

$$\partial_i u_j \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } i \ne j, \ \frac{2}{p} + \frac{3}{q} \le \frac{1}{2} + \frac{3}{2q}, \ 3 < q \le \infty; \tag{11}$$

$$\partial_i u_i \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2}{p} + \frac{3}{q} \le \frac{3}{4} + \frac{3}{2q}, \ 2 < q \le \infty. \tag{12}$$

The limiting case of (11)

$$\partial_i u_j \in L^\infty(0, T; L^3(\mathbb{R}^3))$$

could just be verified by using the specified $r = \frac{7}{3}$ in [5, Lemma 1], see also [9, Theorem 1.1].