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Global regularity of N-dimensional generalized MHD system with anisotropic dissipation and diffusion☆

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ABSTRACT

Motivated by the anisotropic Navier–Stokes equations that have much applications in studying fluids in thin domains, in particular meteorology and oceanography, we study the magnetohydrodynamics system with generalized dissipation and diffusion, considering different exponents of the fractional Laplacians with logarithmic worsening applied to different directions and components of the solution vector fields. The results indicate that it is possible to regularize the flows by anisotropic dissipation, some components in various directions allowed to be below the critical exponents in the expense of others being above.

1. Introduction and statement of theorems

We study the N-dimensional magnetohydrodynamics (MHD) system $N \geq 2$ defined as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla\pi = \nu\Delta u, \\ \dfrac{\partial b}{\partial t} + (u\cdot\nabla)b - (b\cdot\nabla)u = \eta\Delta b, \\ \nabla\cdot u = \nabla\cdot b = 0, \quad (u,b)(x,0) = (u_0,b_0)(x), \end{cases} \tag{1}$$

where $u : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N$, $b : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N$, $\pi : \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}$ are the velocity, magnetic and pressure fields respectively and the parameters $\nu, \eta \geq 0$ represent the kinematic viscosity and diffusivity constants respectively. The system describes the motion of electrically conducting fluids and plays a fundamental role in applied sciences such as astrophysics, geophysics and plasma physics. Let us denote hereafter $\partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x_i}$.

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In both cases $N = 2, 3, \nu, \eta > 0$, the system possesses at least one global L^2-weak solution pair for any initial data pair $(u_0, b_0) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$; moreover, in case $N = 2$, the global existence of strong solution has been shown (cf. [18]). However, the global regularity issue of the solution pair in the case $N \geq 3$ remains open. In fact, because the system at $b \equiv 0$ is reduced to the Navier–Stokes equations (NSE), such an issue seems to be extremely challenging. For example, in case $N = 2$ while the inviscid NSE, the Euler equations, admits a global regularity result (e.g. [13]), it remains unknown whether the same result holds for the MHD system.

To suggest a new approach, the author in [25] studied the generalized MHD system; that is,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla\pi + \nu\Lambda^{2\alpha}u = 0, \\ \partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta\Lambda^{2\beta}b = 0, \end{cases} \tag{2}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ defined through Fourier transform as follows:

$$\mathcal{F}(\Lambda^{2\gamma}f)(\xi) = |\xi|^{2\gamma}\mathcal{F}(f)(\xi), \qquad \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^N} f(x)e^{-ix\cdot\xi}dx.$$

We note that the classical MHD system (1) is the special case of (2), at $\alpha = \beta = 1$. After such a generalization, the author in [25] showed that in case $\nu, \eta > 0$ and $\alpha \geq \frac{1}{2} + \frac{N}{4}$, $\beta \geq \frac{1}{2} + \frac{N}{4}$, (u, b) remains smooth for all time (cf. [35]). This lower bound is related to the fact that, for simplicity, taking $\alpha = \beta = \gamma, \lambda \in \mathbb{R}^+$, if $(u(x,t), b(x,t))$ solves (2), then so does $(u_\lambda(x,t), b_\lambda(x,t))$ where

$$u_\lambda(x,t) = \lambda^{2\gamma-1}u(\lambda x, \lambda^{2\gamma}t), \qquad b_\lambda(x,t) = \lambda^{2\gamma-1}b(\lambda x, \lambda^{2\gamma}t),$$

and precisely when $\alpha = \beta = \frac{1}{2} + \frac{N}{4}$, we have $\|u_\lambda(\cdot,t)\|_{L^2(\mathbb{R}^N)} = \|u(\cdot,\lambda^{2\gamma}t)\|_{L^2(\mathbb{R}^N)}$, $\|b_\lambda(\cdot,t)\|_{L^2(\mathbb{R}^N)} = \|b(\cdot,\lambda^{2\gamma}t)\|_{L^2(\mathbb{R}^N)}$. Reducing this lower bound furthermore below $\frac{1}{2} + \frac{N}{4}$ has become an extremely difficult problem.

Motivated by the recent work in [19,20] on the wave equations and the NSE respectively, the author in [26] improved his own result logarithmically. In short, he generalized the fractional Laplacians furthermore and considered

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla\pi + \nu\mathcal{L}^2 u = 0, \\ \partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta\mathcal{M}^2 b = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \quad (u,b)(x,0) = (u_0,b_0)(x), \end{cases} \tag{3}$$

where $\mathcal{L}$ and $\mathcal{M}$ are defined by

$$\begin{cases} \mathcal{F}(\mathcal{L}f)(\xi) = l(\xi)\mathcal{F}(f)(\xi), & l(\xi) \geq \dfrac{|\xi|^\alpha}{g(\xi)}, \\ \mathcal{F}(\mathcal{M}f)(\xi) = m(\xi)\mathcal{F}(f)(\xi), & m(\xi) \geq \dfrac{|\xi|^\beta}{h(\xi)}, \end{cases} \tag{4}$$

with $\alpha \geq \frac{1}{2} + \frac{N}{4}, \beta > 0, \alpha + \beta \geq 1 + \frac{N}{2}$ and $g, h \geq 1$ are radially symmetric non-decreasing functions such that they satisfy a certain integral condition, and showed that this system has a unique global classical solution (see also [5,8,17,22,27]). Besides logarithmic improvement, these results showed that the exponents of the fractional Laplacians may be shifted toward that of dissipation rather than diffusion, as long as the sum continues to satisfy the lower bound of $\alpha + \beta \geq 1 + \frac{N}{2}$. We also note that e.g. in [10,34] it was shown that the regularity criteria for this system may be shown to rely only on u, dropping conditions on b completely.

In this paper we are concerned with the flexibility of the exponents in each direction and component of u and b. In other words, denoting by $\mathcal{F}_i$ the one-dimensional Fourier transform in the ξ_i-direction, we define Λ_i by

$$\mathcal{F}_i(\Lambda_i^{2\gamma}f)(\xi_i) = |\xi_i|^{2\gamma}(\mathcal{F}_i f)(\xi_i), \quad i = 1, \ldots, N,$$