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Global and blow-up solutions of superlinear pseudoparabolic equations with unbounded coefficient

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ABSTRACT

We investigate positive solutions of pseudoparabolic equations $\partial_t u - \triangle\partial_t u = \triangle u + V(x)u^p$ in $\mathbb{R}^n \times (0,\infty)$, where $p > 1$ and V is a (possibly unbounded or singular) potential. Under some rather weak assumptions on the potential, we establish the existence of solutions, both locally and globally in time, within weighted Lebesgue spaces for the Cauchy problem. Blow-up behavior is also derived using the test function method. As a consequence, we show that if $V = |\,x\,|^\sigma$ where $0 \le \sigma \le \frac{4}{n-2}$ if $n \ge 3$ and $\sigma \in [0,\infty)$ if $n = 1,2$, then the critical exponent of the Cauchy problem is $1 + \frac{\sigma+2}{n}$. This generalizes the result in the case $\sigma = 0$ by Cao et al. (2009).

1. Introduction

We consider nonnegative solutions $u = u(x,t)$ of the Cauchy problem

$$\begin{cases} \partial_t u - \triangle\partial_t u = \triangle u + V(x)u^p & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $p > 1$ is a constant and V, u_0 are non-negative functions. In this work, the "potential" V can be unbounded or singular. Indeed, we put a rather weak assumption that $V \in \mathcal{P}^\sigma$ or $V \in \mathcal{P}_\sigma$ for some $\sigma \in \mathbb{R}$, where

$$\mathcal{P}^\sigma = \{\varphi \in C(\mathbb{R}^n) : \varphi \ge 0, \varphi(x) \lesssim |x|^\sigma \text{ for large } |x|\}, \tag{1.2}$$

$$\mathcal{P}_\sigma = \{\varphi \in C(\mathbb{R}^n \setminus \{0\}) : \varphi \ge 0, \varphi(x) \gtrsim |x|^\sigma \text{ for almost every } x\}. \tag{1.3}$$

Eq. (1.1) is called a semilinear *pseudoparabolic equation* as it possesses some properties similar to heat equations [21–23,25]. Nonlinear pseudoparabolic equations have been employed to model many physical

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systems such as the non-steady flow of second order fluids in one space dimension, seepage of homogeneous fluids through a fissured rock, the heat conduction involving two temperatures, and etc. [1,3,4,9,15,16,19,24]. In recent years, motivated by problems in stochastic partial differential equations, there is a great interest in the study of partial differential equation with unbounded or singular coefficients.

When there is no viscous term $\triangle\partial_t u$, (1.1) is reduced to the Cauchy problem of semilinear heat equation

$$\partial_t u = \triangle u + V(x)u^p \quad \text{in } \mathbb{R}^n \times (0,\infty), \qquad u|_{t=0} = u_0 \geq 0 \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

For the case $V = 1$, this problem has now become classic by the fundamental and inspiring works of Fujita [6], Hayakawa [8], Kobayashi et al. [14], Aronson et al. [2], and Weissler [26]. In summary, the Cauchy problem has a unique solution for each $u_0 \in L^\infty(\mathbb{R}^n)$; moreover, if $1 < p \leq 1 + \frac{2}{n}$ the solution always blows up in a finite time unless $u_0 \equiv 0$, whereas if $p > 1 + \frac{2}{n}$, the solution can be global if u_0 is sufficiently small or it can be blowing-up if u_0 is sufficiently large. The number $p_c = 1 + \frac{2}{n}$ is called the (*Fujita*) *critical exponent* of the Cauchy problem. For the case of variable potentials V, the problem (1.4) was studied by Pinsky [20] in 1997 under the assumption that $V(x) \sim |x|^\sigma$ if $\sigma > -2$ or $0 \lneqq V(x) \lesssim |x|^{-2}$. By deriving a priori pointwise and integral estimates involving the heat kernel, the author was able to show that the critical exponent for the Cauchy problem in this case is

$$p_{c,\sigma} = \begin{cases} 1 + \dfrac{\sigma+2}{n} & \text{if } n \geq 2,\ \sigma \in (-2,\infty);\ \text{or } n = 1,\ \sigma \in (-1,\infty), \\ 2 & \text{if } n = 1,\ \sigma \in [-2,-1]. \end{cases}$$

Pseudoparabolic equations are quite new. The linear theory had been developed around 60–80's, see for instance, [25,23,7]. Some important results related to (1.1) are sketched in the next section. In the case $V \equiv 1$, it was shown by Y. Cao et al. [5] in 2009 that the critical exponent of (1.1) is $p_c = 1 + \frac{2}{n}$. Their arguments can be directly adapted to get the same result in the case of variable but *bounded* potentials. Up to the author knowledge, the investigation of the problem (1.1) having *unbounded* or *singular* potentials is still open. This constitutes the main goal of the present work. Unlike the heat equation, the complicated fundamental solution for (1.1) makes it impossible to derive a priori pointwise and integral estimates similar to [20]. In addition, the unboundedness or singular behavior of V prevent us from carrying out the analysis in [5]. In fact, if V is unbounded or singular, then the propagator operator for mild solutions is discontinuous on $L^\infty(\mathbb{R}^n)$, the spatial space used in [5,20], and there is no comparison principle. The novelty of this work is that, by replacing the usual $L^\infty(\mathbb{R}^n)$ spaces with the *weighted Lebesgue spaces* $L^{\infty,a}(\mathbb{R}^n)$, we can overcome all the difficulties above.

Let us describe our main results. In Section 2, we introduce notation, elementary results, and some linear theory for the pseudoparabolic equation (1.1). The *Bessel potential operator* $\mathcal{B} = (I - \triangle)^{-1}$ and the *Green operator* $\mathcal{G}(t) = e^{t\mathcal{B}\triangle}$ $(t > 0)$ from the linear theory together with their well-known estimates are recalled in this section. We consider mainly, in this work, mild solutions with values in weighted Lebesgue spaces $L^{\infty,a}(\mathbb{R}^n)$. Thus, the boundedness of $\mathcal{B}$ and $\mathcal{G}(t)$ on $L^{\infty,a}(\mathbb{R}^n)$ are crucial. The boundedness of $\mathcal{B}$ is more or less well-known and is sketched in Section 2. Our first result is the boundedness of $\mathcal{G}(t)$ on the weighted Lebesgue spaces $L^{q,a}(\mathbb{R}^n)$ for any $1 \leq q \leq \infty$ and $a \in \mathbb{R}$, and an interpolation estimate. Both are proved in Section 3. We use the (Fourier) phase space analysis guiding by [11], where the case $q = 1$, $a \in [0,1)$ is treated. The interpolation estimate will be crucial in establishing the existence of global solutions.

The second result is the existence and uniqueness of local (in time) solutions in Section 4. We show that if $p > 1$, $V \in \mathcal{P}^\sigma$ with $\sigma \in \mathbb{R}$, and $a \in \mathbb{R}$, then the Cauchy problem (1.1) has a unique mild solution $u \in C\left([0,T); L_C^{\infty,a}(\mathbb{R}^n)\right)$ for some $T > 0$ provided

$$a \geq \frac{\sigma}{p-1}.$$

It is interesting to note that there is a lower bound for the order a of the weight and this lower bound converges to ∞ as $p \to 1^+$ if $\sigma > 0$. Equivalently, the solutions to the Cauchy problem with $\sigma > 0$ have to