Nonlinear Analysis 121 (2015) 19–35

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Well-posedness of the Cauchy problem for a fourth-order thin film equation via regularization approaches

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ARTICLE INFO

Communicated by S. Carl

MSC:
35K65
35A09
35G20
35K25

Keywords:
Thin film equation
The Cauchy problem
Finite interfaces
Oscillatory sign-changing behaviour
Analytic ε-regularization
Uniqueness

ABSTRACT

This paper is devoted to some aspects of well-posedness of the Cauchy problem (the CP, for short) for a quasilinear degenerate fourth-order parabolic *thin film equation* (the TFE-4)

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{0.1}$$

where $n > 0$ is a fixed exponent, with bounded smooth compactly supported initial data. Dealing with the CP (for, at least, $n \in (0, \frac{3}{2})$) requires introducing classes of infinitely changing sign solutions that are oscillatory close to finite interfaces. The main goal of the paper is to detect proper solutions of the CP for the degenerate TFE-4 by uniformly parabolic analytic ε-regularizations at least for values of the parameter n sufficiently close to 0.

Firstly, we study an analytic "homotopy" approach based on *a priori* estimates for solutions of uniformly parabolic analytic ε-regularization problems of the form

$$u_t = -\nabla \cdot (\phi_\varepsilon(u) \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+,$$

where $\phi_\varepsilon(u)$ for $\varepsilon \in (0, 1]$ is an analytic ε-regularization of the problem (0.1), such that $\phi_0(u) = |u|^n$ and $\phi_1(u) = 1$, using a more standard classic technique of passing to the limit in integral identities for weak solutions. However, this argument has been demonstrated to be non-conclusive, basically due to the lack of a complete optimal estimate-regularity theory for these types of problems.

Secondly, to resolve that issue more successfully, we study a more general similar analytic "homotopy transformation" in both the parameters, as $\varepsilon \to 0^+$ and $n \to 0^+$, and describe *branching* of solutions of the TFE-4 from the solutions of the notorious *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

which describes some qualitative oscillatory properties of CP-solutions of (0.1) for small $n > 0$ providing us with the uniqueness of solutions for the problem (0.1) when n is close to 0.

Finally, *Riemann-like problems* occurring in a boundary layer construction, that occur close to nodal sets of the solutions, as $\varepsilon \to 0^+$, are discussed in other to get uniqueness results for the TFE-4 (0.1).

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http://dx.doi.org/10.1016/j.na.2014.08.002
0362-546X/

1. Introduction: the Cauchy and free boundary problems for the TFE-4

1.1. Main model and their applications

In this paper, we study some aspects of well-posedness of the Cauchy problem (the CP) for a nowadays well-known fourth-order quasilinear evolution equation of parabolic type, called the *thin film equation* (the TFE-4), with an exponent $n > 0$,

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

with bounded, sufficiently smooth, and compactly supported initial data u_0 (not necessarily positive) with an arbitrary dimension $N \geq 1$. Note that these initial conditions could be relaxed, (for example $u_0 \in L^1 \cap L^\infty$) however, it is not the purpose of this work to analyse the problem from the perspective of different possible initial conditions.

Eq. (1.1) arises in numerous physical related areas with applications in thin film, lubrication theory, and in several other hydrodynamic-type problems. In particular, those equations model the dynamics of a thin film of viscous fluid, as the spreading of a liquid film along a surface, where u stands for the height of the film. Then clearly assuming $u \geq 0$ naturally leads to a *free boundary problem* (an FBP) setting; see below. Specifically, when $n = 3$, we are dealing with a problem in the context of lubrication theory for thin viscous films that are driven by surface tension and when $n = 1$ with Hele-Shaw flows. However, in this work, we are considering solutions of changing sign. Such solutions can have some biological motivations [1], to say nothing of general PDE theory, where the CP-settings were always key.

1.2. Main results

As a more successful approach, among others, to clarify the well-posedness of the CP, we perform an analytic "homotopic" approach from our original Eq. (1.1) to an equation from which we can extract information about the solutions of Eq. (1.1),

Namely, we develop a *homotopic deformation* from the TFE-4 (1.1) to the classic and well-known *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

It is well-known that, for any smooth compactly supported data u_0, satisfying the natural "growth condition at infinity"

$$u_0 \in L^2_{\rho^*}(\mathbb{R}^N), \quad \text{where } \rho^*(y) = e^{-a|y|^{4/3}}, \ a = \text{const.} > 0 \text{ small},$$

the *bi-harmonic* equation (1.2) admits a unique classic solution given by the convolution Poisson-type integral,

$$\tilde{u}(x,t) = b(x,t) * u_0(x) \equiv t^{-\frac{N}{4}} \int_{\mathbb{R}^N} F\left((x-z)t^{-\frac{1}{4}}\right) u_0(z)\,dz, \tag{1.3}$$

where $b(x,t)$ is the fundamental solution

$$b(x,t) = t^{-\frac{N}{4}} F(y), \quad y = \frac{x}{t^{1/4}}, \tag{1.4}$$

of the operator $\frac{\partial}{\partial t} + \Delta^2$. The oscillatory rescaled kernel $F(y)$ is the unique solution of the linear elliptic problem

$$\mathbf{B}F \equiv -\Delta_y^2 F + \frac{1}{4} y \cdot \nabla_y F + \frac{N}{4} F = 0 \quad \text{in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} F(y)\,dy = 1. \tag{1.5}$$

First regularization. Thus, firstly, we perform a homotopic deformation assuming that $n > 0$ is a sufficiently small and *fixed* exponent. We are then actually talking about some "homotopic classes" (understood here not in the classic sense from degree operator theory) of degenerate parabolic PDEs.

More precisely, we say that the TFE (1.1) is "homotopic to the linear PDE (1.2)" if there exists a family of uniformly parabolic equations (a *homotopic deformation*) with a coefficient

$$\phi_\varepsilon(u) > 0 \quad \text{analytic in both variables } u \in \mathbb{R} \text{ and } \varepsilon \in (0,1],$$

with unique analytic solutions $u_\varepsilon(x,t)$ of the problem

$$u_t = -\nabla \cdot (\phi_\varepsilon(u) \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

such that

$$\phi_1(u) = 1, \quad \text{and} \quad \phi_\varepsilon(u) \to |u|^n \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on compact subsets}, \tag{1.7}$$

so that $u(x,t)$ can be approximated by $u_\varepsilon(x,t)$ as $\varepsilon \to 0^+$.

A possible and quite natural homotopic path (to be used in this work) is

$$\phi_\varepsilon(u) := \varepsilon^n + (1-\varepsilon)(\varepsilon^2 + u^2)^{\frac{n}{2}}, \quad \varepsilon \in (0,1]. \tag{1.8}$$

Then, indeed, the non-degenerate uniformly parabolic equation (1.6) admits a unique (at least, locally in time) classic solution $u = u_\varepsilon(x,t)$, which is an analytic function in all the three variables x, t, and ε, just using classic parabolic theory [2,3], for any $\varepsilon \in (0,1]$.