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Elliptic inequalities with multi-valued operators: Existence, comparison and related variational–hemivariational type inequalities

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ABSTRACT

We study multi-valued elliptic variational inclusions in a bounded domain $\Omega \subset \mathbb{R}^N$ of the form

$$u \in K : 0 \in Au + \partial I_K(u) + \mathcal{F}(u) + \mathcal{F}_\Gamma(u),$$

where A is a second order quasilinear elliptic operator of Leray–Lions type, K is a closed convex subset of some Sobolev space, I_K is the indicator function related to K, and ∂I_K denoting its subdifferential. The lower order multi-valued operators $\mathcal{F}$ and $\mathcal{F}_\Gamma$ are generated by multi-valued, upper semicontinuous functions $f : \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ and $f_\Gamma : \Gamma \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, respectively, with $\Gamma \subset \partial\Omega$. Our main goals are as follows: First we provide an existence theory for the above multi-valued variational inequalities. Second, we establish an enclosure and comparison principle based on appropriately defined sub–supersolutions, and prove the existence of extremal solutions. Third, by means of the sub–supersolution method provided here, we are going to show that rather general classes of variational–hemivariational type inequalities turn out to be only subclasses of the above general multi-valued elliptic variational inequalities, which in a way fills a gap in the current literature where these kind of problems are studied independently. Finally, the existence of extremal solutions will allow us to deal with classes of multi-valued function f and f_Γ that are neither lower nor upper semicontinuous, which in turn will provide a tool to obtain existence results for variational–hemivariational type inequalities whose Clarke's generalized directional derivative may, in addition, discontinuously depend on the function we are looking for. This paper, though of surveying nature, provides an analytical framework that allows to present in a unifying way and to extend a number of recent results due to the authors.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $\Gamma \subset \partial\Omega$ be a relatively open subset, and denote $\Gamma_0 = \partial\Omega \setminus \Gamma$ such that

$$\partial\Omega = \Gamma \cup \Gamma_0.$$

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Let $V = W^{1,p}(\Omega)$, $1 < p < +\infty$, denote the usual Sobolev space with its dual space V^*, and let $V_0 \subset V$ be the subspace of V defined by

$$V_0 = \{u \in V : \tau u|_{\Gamma_0} = 0\},$$

where $\tau : V \to L^{\tilde{q}}(\partial\Omega)$ stands for the trace operator which is known to be linear and compact for $1 \le \tilde{q} < \tilde{p}^*$ where $\tilde{p}^*$ is the trace critical Sobolev exponent to p, see [15]. In what follows we are going to use the following notation for the restrictions of τu

$$\gamma u := \tau u|_{\Gamma}, \qquad \gamma_0 u := \tau u|_{\Gamma_0}.$$

Note that $V = V_0$ in case that $\Gamma = \partial\Omega$, and in case that $\Gamma = \emptyset$ we have $\Gamma_0 = \partial\Omega$ and thus V_0 is the subspace of V whose functions have generalized homogeneous boundary values. Let $K \subset V_0$ be a closed, convex subset. In this paper we provide an analytical framework to study existence and enclosure results for the following multi-valued elliptic variational inclusion

$$u \in K : 0 \in Au + \partial I_K(u) + \mathcal{F}(u) + \mathcal{F}_\Gamma(u), \tag{1.1}$$

where A is a second order quasilinear elliptic differential operator in divergence form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right), \tag{1.2}$$

and I_K is the indicator function related to K with ∂I_K denoting its subdifferential. By $\mathcal{F}$ and $\mathcal{F}_\Gamma$ we denote lower order multi-valued operators generated by multi-valued functions $(x, s) \mapsto f(x, s)$, $(x, s) \mapsto f_\Gamma(x, s)$ with $f : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}\setminus\{\emptyset\}$ and $f_\Gamma : \Gamma \times \mathbb{R} \to 2^{\mathbb{R}}\setminus\{\emptyset\}$, respectively, which are only supposed to be upper semicontinuous with respect to their second argument s, and measurable. To make the notion of solution of (1.1) more precise, let us introduce the multi-valued Nemytskij operators F and F_Γ associated with the multi-valued functions f and f_Γ, respectively, by

$$\begin{aligned} F(u) &= \{\eta : \Omega \to \mathbb{R} : \eta \text{ is measurable in } \Omega \text{ and } \eta \in f(\cdot, u)\}, \\ F_\Gamma(\gamma u) &= \{\zeta : \Gamma \to \mathbb{R} : \zeta \text{ is measurable on } \Gamma \text{ and } \zeta \in f_\Gamma(\cdot, \gamma u)\} \end{aligned} \tag{1.3}$$

where $\eta(x) \in f(x, u(x))$ for a.a. $x \in \Omega$, and $\zeta(x) \in f_\Gamma(x, \gamma u(x))$ for a.a. $x \in \Gamma$. Let $q \in [1, p^*)$ and $\tilde{q} \in [1, \tilde{p}^*)$ with p^* and $\tilde{p}^*$ being respectively the critical Sobolev exponent and trace critical Sobolev exponent to p,

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \le p, \end{cases}$$

and

$$\tilde{p}^* = \begin{cases} \dfrac{(N-1)p}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \le p. \end{cases}$$

Under certain assumptions to be specified later, the multi-valued Nemytskij operators $F : L^q(\Omega) \to 2^{L^{q'}(\Omega)}$ and $F_\Gamma : L^{\tilde{q}}(\Gamma) \to 2^{L^{\tilde{q}'}(\Gamma)}$ are well defined, where q' and $\tilde{q}'$ denote the Hölder conjugates to q and $\tilde{q}$ respectively, i.e., $1/q + 1/q' = 1/\tilde{q} + 1/\tilde{q}' = 1$. For $1 \le q < p^*$, the embedding $i : V_0 \hookrightarrow L^q(\Omega)$ is compact, and thus its adjoint operator $i^* : L^{q'}(\Omega) \hookrightarrow V_0^*$ is compact as well. Similarly, for $\tilde{q} \in [1, \tilde{p}^*)$, the trace operator $\gamma : V_0 \to L^{\tilde{q}}(\Gamma)$ as well as its adjoint operator $\gamma^* : L^{\tilde{q}'}(\Gamma) \to V_0^*$ is compact. Hence, the composed multi-valued operators

$$\mathcal{F} = i^* \circ F \circ i : V_0 \to 2^{V_0^*} \quad \text{and} \quad \mathcal{F}_\Gamma = \gamma^* \circ F_\Gamma \circ \gamma : V_0 \to 2^{V_0^*} \tag{1.4}$$

are well defined as well, and given by

$$\begin{aligned} \mathcal{F}(u) &= \{\eta^* = i^*\eta \in V_0^* : \eta \in F(u)\}, \quad \text{with } \langle \eta^*, \varphi\rangle = \int_\Omega \eta\,\varphi\,dx,\ \forall\,\varphi \in V_0, \\ \mathcal{F}_\Gamma(u) &= \{\zeta^* = \gamma^*\zeta \in V_0^* : \zeta \in F_\Gamma(\gamma u)\}, \quad \text{with } \langle \zeta^*, \varphi\rangle = \int_\Gamma \zeta\,\gamma\varphi\,d\Gamma,\ \forall\,\varphi \in V_0, \end{aligned} \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of V_0^* and V_0. Now we can precisely state the notion of solution of the multi-valued variational inequality (1.1) (MVI for short).

Definition 1.1. A function $u \in K \subset V_0$ is a solution of the MVI (1.1), if there exist $q \in [1, p^*)$, $\tilde{q} \in [1, \tilde{p}^*)$, $\eta \in L^{q'}(\Omega)$, and $\zeta \in L^{\tilde{q}'}(\Gamma)$ such that

$$\begin{cases} \eta \in F(u), \quad \zeta \in F_\Gamma(\gamma u), \\ \langle Au, v - u\rangle + \displaystyle\int_\Omega \eta\,(v - u)\,dx + \int_\Gamma \zeta\,(\gamma v - \gamma u)\,d\Gamma \ge 0, \quad \forall\, v \in K. \end{cases} \tag{1.6}$$

Note that $\eta \in F(u)$ means $\eta(x) \in f(x, u(x))$ for a.a. $x \in \Omega$, and $\zeta \in F_\Gamma(\gamma u)$ means $\zeta(x) \in f_\Gamma(x, \gamma u(x))$ for a.a. $x \in \Gamma$.