Nonlinear Analysis 121 (2015) 206–222

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Harnack inequalities for double phase functionals

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ARTICLE INFO

Communicated by S. Carl

To Enzo Mitidieri, with friendship

Keywords:
Regularity
Harnack inequality
Degenerate problems

ABSTRACT

We prove a Harnack inequality for minimisers of a class of non-autonomous functionals with non-standard growth conditions. They are characterised by the fact that their energy density switches between two types of different degenerate phases.

1. Introduction and results

In this paper we complete the study of the low order regularity properties of minima of a class of functionals with non-standard growth conditions. They are basically characterised by the fact of having the energy density switching between two different types of degenerate behaviours, according to the size of a "modulating coefficient" $a(\cdot)$ that determines the "phase". Specifically, we consider a family of functionals whose model is given by the following one:

$$\mathcal{P}_{p,q}(w,\Omega) := \int_\Omega \left(|Dw|^p + a(x)|Dw|^q\right) dx \tag{1.1}$$

where $1 < p \le q$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with $n \ge 2$. In this paper the function $a(\cdot)$ will always be assumed to be bounded and non-negative. In the standard case $p = q$ the functional in question has standard p-polynomial growth and the regularity theory of minimisers is by now well-understood; see for instance [26,19,29]. The case $p < q$ falls in the realm of functionals with non-standard growth conditions of (p, q) type, as initially defined and studied by Marcellini [27,28]. These are general functionals of the type

$$W^{1,1}(\Omega) \ni w \longmapsto \mathcal{F}_{p,q}(w,\Omega) := \int_\Omega F(x, w, Dw)\, dx, \quad \Omega \subset \mathbb{R}^n, \tag{1.2}$$

where the integrand $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,\infty)$ is a Carathéodory function satisfying bounds of the type

$$|z|^p \lesssim F(x, v, z) \lesssim |z|^q + 1 \quad 1 < p < q \tag{1.3}$$

whenever $(x, v, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Indeed, the energy density

$$H_{p,q}(x,z) := |z|^p + a(x)|z|^q \tag{1.4}$$

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http://dx.doi.org/10.1016/j.na.2014.11.001
0362-546X/

of the functional $\mathcal{P}_{p,q}$ in (1.1) exhibits a polynomial growth of order q with respect to the gradient variable z when $a(x) > 0$ (this is the "(p, q)-phase"), while on the phase transition zero set $\{a(x) = 0\}$ the growth is at rate p (this is the "p-phase"). Therefore, from a global viewpoint, also the functional $\mathcal{P}_{p,q}$ satisfies (1.3) and therefore falls in the realm of those with (p, q)-growth conditions. Now, while in the case of an autonomous energy density of the type $F(x, w, Dw) \equiv F(Dw)$ the regularity theory of minima of functionals with (p, q)-growth conditions is by now well-understood (see for instance [4,5,27,13,24,29]), the case of non-autonomous integrals is still very much open and indeed new phenomena appear, which are directly linked to the specific structure of the functional. In this paper we are interested in functionals whose structure exhibits a phase transition as in (1.1). The functional $\mathcal{P}_{p,q}$ belongs to a family of variational integrals introduced by Zhikov [32,35] in order to produce models for strongly anisotropic materials. They intervene in Homogenisation theory and Elasticity, where the coefficient $a(\cdot)$ for instance dictates the geometry of a composite made by two different materials. They can also be used in order to provide new examples of Lavrentiev phenomenon [33,34]. For the functional $\mathcal{P}_{p,q}$ a very sharp interaction occurs between the size of the phase transition, measured by the distance between p and q, and the regularity of the coefficient $a(\cdot)$, as initially shown in [14,16]. There, for every $\varepsilon > 0$, it has been shown the existence of a coefficient function $a(\cdot) \in C^{0,\alpha}$, and of exponents p, q satisfying

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon, \tag{1.5}$$

such that there exist *bounded* minimisers of $\mathcal{P}_{p,q}$ whose set of essential discontinuity points has Hausdorff dimension larger than $n-p-\varepsilon$. In other words, minimisers can be almost as bad as any other $W^{1,p}$-function. Regularity assertions are instead more recent. In [6] the last two named authors have shown that the conditions

$$0 \le a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad q \le p + \alpha \tag{1.6}$$

for some $\alpha \in (0, 1]$, are sufficient in order to prove local Hölder continuity of locally bounded minimisers of the functional $\mathcal{P}_{p,q}$. The numerology displayed in (1.5) shows that the conditions in (1.6) are sharp. It is worthwhile to mention that the results of [6] cover more general functionals than $\mathcal{P}_{p,q}$ and that further conditions, this time involving also the ambient dimension n, eventually allow to conclude that any local minimiser is locally bounded. We shall come back on these points in Remark 1.3.

Starting from the Hölder continuity result of [6] and inspired by what happens in the case of functionals with standard polynomial growth ($p = q$), we now wonder if a suitable Harnack inequality holds for non-negative minimisers. We show here that the answer to this question is positive and that Harnack inequality holds in the case of functionals with measurable coefficients, but still encoding the peculiar structure of $\mathcal{P}_{p,q}$, in terms of growth conditions. We indeed consider functionals of the type in display (1.2) where the energy density $F(\cdot)$ is only assumed to be a Carathéodory function satisfying the bounds

$$\nu \le \frac{F(x, v, z)}{H_{p,q}(x, z)} \le L \tag{1.7}$$

whenever $z \in \mathbb{R}^n \setminus \{0\}$, $v \in \mathbb{R}$ and $x \in \Omega$, where $0 < \nu \le 1 \le L$; $H_{p,q}(\cdot)$ has been defined in (1.4). In this setting we recall that a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a local minimiser of the functional in (1.2) if and only if $F(x, u, Du) \in L^1_{\text{loc}}(\Omega)$ and the minimality condition

$$\int_{\text{supp}\,(u-w)} F(x, u, Du)\, dx \le \int_{\text{supp}\,(u-w)} F(x, w, Dw)\, dx$$

is satisfied whenever $w \in W^{1,1}_{\text{loc}}(\Omega)$ is such that $\text{supp}\,(u - w) \Subset \Omega$. Since we are assuming (1.7), and $F(x, u, Du) \in L^1_{\text{loc}}(\Omega)$, without loss of generality we may assume that all $W^{1,1}_{\text{loc}}$-minimisers will automatically be in $W^{1,p}_{\text{loc}}(\Omega)$, since the lower bound in (1.3) will always be in force for the functionals we are going to consider. Our first result is now the following:

Theorem 1.1. *Let $u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^{\infty}_{\text{loc}}(\Omega)$ be a non-negative local minimiser of the functional $\mathcal{F}_{p,q}$, defined in (1.2), under the assumptions (1.7), (1.6) and with $p < n$. Then for every ball B_R with $B_{9R} \subset \Omega$ there exists a constant c, depending on $n, p, q, \nu, L, \alpha, [a]_{C^{0,\alpha}(\Omega)}, \|u\|_{L^\infty(B_{9R})}$ and $\text{diam}\,(\Omega)$, such that*

$$\sup_{B_R} u \le c \inf_{B_R} u$$

holds.

In the case $p > n$ minimisers are automatically locally bounded by the Sobolev embedding theorem, so that assuming $u \in L^{\infty}_{\text{loc}}(\Omega)$ is superfluous. The same happens when $p = n$ by means of the results of [9], see Remark 1.3. On the other hand, as already noticed in [6,7], when $p > n$ the condition in (1.6) can be relaxed, see also Remark 1.3. Indeed, we shall consider

$$0 \le a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad \frac{q}{p} \le 1 + \frac{\alpha}{n}. \tag{1.8}$$