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An invitation to modern group analysis of differential equations

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ABSTRACT

The purpose of this review paper is two-fold. First, we point out one more time the importance and usefulness of the S. Lie theory of symmetries of differential equations. In particular some illustrative examples are presented. Then, we discuss the Noetherian approach to Rellich–Pohozaev type identities.

1. Introduction

To begin with, we would like to ask a simple question:

Why does a differential equation admit an integrating factor?

According to our personal experience and contacts, unfortunately, even some distinguished contemporary specialists in differential equations cannot satisfactorily respond to this question. The answer is given by the celebrated Sophus Lie theory of symmetries of differential equations. Grosso modo, the symmetries of a differential equation transform solutions of the equation to other solutions. One of the main benefits of this theory is that by following a completely algorithmic procedure one is able to determine the symmetries of a differential equation or systems of differential equations. The Lie point symmetries comprise a structural property of the equation—in essence they are the DNA of an equation. The knowledge of the symmetries of an equation enables one to use them for a variety of purposes, from obtaining analytical solutions and reducing its order to finding of integrating factors and conservation laws. In fact, many, if not all, of the different empirical methods for solving ordinary differential equations (ODE) we have learned from standard courses at the undergraduate level emerge from a symmetry. For instance, having at our disposal a Lie point symmetry of a first order ODE, we can immediately get explicitly an integrating factor by a formula obtained by S. Lie, answering the question raised above. In this regard we would like to remind the words of Nail Ibragimov that "one of the most remarkable achievements of Lie was

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the discovery that the majority of known methods of integration of ordinary differential equations, which seemed up to that time artificial and internally disconnected, could be derived in a unified manner using the theory of groups. Moreover, Lie provided a classification of all ordinary differential equations of arbitrary order in terms of the symmetry groups they admit, and thus described the whole collection of equations for which integration or lowering of the order could be effected by group-theoretical methods. However, these and other very valuable results he obtained could not for a long time be widely disseminated and remained known to only a few. It could be said that this is the state of affairs today with methods of solution of problems of mathematical physics: many of these are of a group-theoretical nature, but are presented as a result of a lucky guess'' [25].

Furthermore, even the knowledge of a trivial solution of the equation can be used for creating nontrivial solutions by using the equation's symmetries. And all these are due to the rich underlining algebraic structure of Lie groups and Lie algebras with which we give flesh to the symmetries of a differential equation. Another important characteristic of the symmetry method is that in some situations the symmetries of an equation may indicate that it can be transformed to a linear equation, to be linearized. In addition, its symmetries provide the means to construct the needed transformation. A strong indication for that is the existence of an infinite dimensional Lie algebra, [6]. In Section 3 an example of this kind is presented and discussed.

If the considered problem has a variational structure then it makes sense to find out those symmetries which leave the action functional invariant. Such symmetries are called variational and divergence symmetries. Their importance is well known: they determine conservation laws via Noether's Theorem [6,35,36]. Thus, with the Noether symmetries in hand, it is immediate to establish the conservation laws. We would also like to point out the well-known fact that conservation laws in mathematical analysis are used for obtaining a priori estimates on solutions of the considered equations and systems, which themselves may lead to existence and nonexistence results.

If the studied equation is of odd order or, more generally, if the problem does not possess variational structure, one can still use the recent new conservation theorem of Nail Ibragimov and the direct construction method by George Bluman and collaborators. (However here we shall not go into discussion of the last two approaches.)

The corner stone of the Noether Theorem is an identity satisfied by the Lagrangian and some arbitrary smooth functions. This identity was called in [23] Noether Identity in honor of Emmy Noether. As pointed out in [23] (see also [24]) it is clear that this identity makes the proof of the Noether Theorem [35,36] purely algebraic and very simple. The Noether Identity can be used to generate integral identities as we shall briefly discuss below.

Now it is well-known that integral identities play an important role in the theory of functions and differential equations. Basically there are two kinds of such identities which are most frequently used, namely, Pohozaev type identities [39,40] and Rellich type identities [43]. It is important to observe that the Pohozaev identities are satisfied by solutions of Dirichlet boundary problems or other semilinear or quasilinear problems, while the Rellich identities concern functions which belong to certain functions spaces without any reference to other relations which they may satisfy like differential equations or boundary conditions. For this reason Rellich's Identity is an important tool for obtaining, among other things, a priori bounds of solutions for semilinear Hamiltonian elliptic systems [16], existence and nonexistence results (see for instance [29,31,30, 32,18]) and sharp Hardy type inequalities [33].

In a series of papers [11–14] Enzo Mitidieri and the author of the present text have proposed a unified approach to both Rellich and Pohozaev type identities. This research has initiated with the paper [11] in which a Noetherian approach to Pohozaev's identities is devised and developed. Its essential point is that the latter can be obtained from the Noether Identity [23,24] after integration, application of the divergence theorem and taking into account the corresponding equations (or systems) and boundary conditions. In this procedure one chooses critical values of the involved parameters. How to do this is explained in [12]. In a subsequent work [13] we have applied this method to some semilinear partial differential equations and systems on Riemannian manifolds. For this purpose we have employed conformal Killing vector fields, which are related to Noether symmetries of critical differential equations. In fact, this generalizes the original idea of Pohozaev [39] who has made use of the radial vector filed $X = r\frac{\partial}{\partial r} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$. We have also obtained in [13] a Rellich type identity on manifolds following the argument in [29,31].

The goal of the work [14] is the observation that the Rellich type identities for functions on Riemannian manifolds can be generated by integration of the Noether Identity for appropriate differential functions (see below for the latter notions). E.g., Rellich's Identity for a function [43] or for a pair of functions [29,31] can be obtained in this way using a well-known Lagrangian and the radial vector field X, mentioned above, determining a dilatation in $\mathbb{R}^n$. One can see that both Pohozaev's and Rellich's identities come from Noether's Identity. In Section 2 some details of the Noetherian approach to Rellich–Pohozaev identities are presented following closely the exposition in [11–14], as we have already done in some parts of this introduction. We think it is worth pointing out this one more time since commonly the specific form of such identities for each concrete problem is obtained by using certain ad hoc procedures, e.g. multiplying the considered equations by some appropriate functions, integrating by parts and then summing up the results. It is also worth observing that there is a certain kind of interplay between the integral identities of Pohozaev–Rellich type, Hardy–Sobolev Inequalities, Liouville type theorems, existence of conformal Killing vector fields and divergence symmetries of nonlinear Poisson equations on Riemannian manifolds.

We would also like to observe that the importance of group classification of differential equations was first emphasized by L.V. Ovsyannikov in 1950s–1960s, when he and his school began a systematic research program of successfully applying modern group analysis methods to wide range of physically important problems.