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Nonlinear elliptic inequalities with gradient terms on the Heisenberg group

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Dedicated to Professor Enzo Mitidieri on the occasion of his 60th birthday, with great feelings of esteem and affection

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ABSTRACT

In this paper we give sufficient conditions both for existence and for nonexistence of nontrivial nonnegative entire solutions of nonlinear elliptic inequalities with gradient terms on the Heisenberg group. The picture is completed with the presentation of a uniqueness result which is, as far as we know, the first attempt for general equations with gradient terms on the Heisenberg group.

1. Introduction

In this paper we first study existence and uniqueness of nonnegative nontrivial radial stationary entire solutions u of

$$\Delta^{\varphi}_{\mathbb{H}^m} u = f(u)\ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}), \tag{1.1}$$

where $\Delta^{\varphi}_{\mathbb{H}^m} u$ is the φ-Laplacian on the Heisenberg group $\mathbb{H}^m$, whose rigorous definition will be given in Section 2, and then for

$$\Delta^{\varphi}_{\mathbb{H}^m} u \geq f(u)\ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) \tag{1.2}$$

Liouville type theorems, that is non-existence of nonnegative nontrivial entire solutions u.

The operator $\Delta^{\varphi}_{\mathbb{H}^m}$ includes as main prototype the well known *Kohn–Spencer Laplacian in* $\mathbb{H}^m$. Moreover, f, ℓ and φ *satisfy throughout the paper*

$$f,\ \ell \in C(\mathbb{R}^+_0), \quad f > 0 \quad \text{and} \quad \ell > 0 \text{ in } \mathbb{R}^+, \tag{$\mathscr{H}$}$$

$$\varphi \in C(\mathbb{R}^+_0) \cap C^1(\mathbb{R}^+), \quad \varphi(0) = 0, \quad \varphi' > 0 \quad \text{in } \mathbb{R}^+, \quad \lim_{s\to\infty} \varphi(s) = \varphi(\infty) = \infty. \tag{ϕ}$$

In particular, in the case of the p-Laplacian, that is when $\varphi(s) = s^{p-1},\ p > 1$, we simply write $\Delta^{p}_{\mathbb{H}^m} u$.

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Since 1957 it is well known that for semilinear coercive inequalities in the Euclidean setting, existence of solutions, as well as nonexistence, involves the Keller–Osserman condition, cfr. [15,23]. For further generalization to quasilinear inequalities, possibly with singular of degenerate weights, we refer to [7–10,21,22]. The first result in this direction, but in the Heisenberg group setting, can be found in [17,2]. Recently, this has been extended to the Carnot groups in [1], adding further restrictions due to the presence of a new term which arises since the norm is not ∞-harmonic in that setting.

Since we are interested in nonnegative entire solutions of elliptic coercive inequalities in all the space, as in [10,17,2] we make use of an appropriate *generalized Keller–Osserman condition for inequality* (1.2). To this aim we also *assume throughout the paper that*

$$\int_{0^+} \frac{t\varphi'(t)}{\ell(t)}\,dt < \infty, \qquad \int^{\infty} \frac{t\varphi'(t)}{\ell(t)}\,dt = \infty, \tag{ϕL}$$

holds. Thus *the function* $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *given by*

$$K(s) = \int_0^s \frac{t\varphi'(t)}{\ell(t)}\,dt \tag{1.3}$$

is a C^1*-diffeomorphism from* $\mathbb{R}_0^+$ *to* $\mathbb{R}_0^+$, with

$$K'(s) = \frac{s\varphi'(s)}{\ell(s)} > 0 \quad \text{in } \mathbb{R}^+, \tag{1.4}$$

thanks to (ϕ) and ($\mathscr{H}$). Consequently K has increasing inverse $K^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and denoting by $F(s) = \int_0^s f(t)\,dt$ we say that *the generalized Keller–Osserman condition holds for* (1.2) if

$$\int^{\infty} \frac{ds}{K^{-1}(F(s))} < \infty. \tag{KO}$$

If $\ell \equiv 1$, then K coincides with the function

$$H(s) = s\varphi(s) - \int_0^s \varphi(t)\,dt, \quad s \geq 0,$$

which represents the Legendre transform of $\Phi(s) = \int_0^s \varphi(t)\,dt$ for all $s \in \mathbb{R}_0^+$. Furthermore, in the case of the p-Laplacian, $H(s) = (p-1)s^p/p$, so that if $\ell \equiv 1$, then (KO) reduces to the well known Keller–Osserman condition for the p-Laplacian, that is $\int^{\infty} F(s)^{-1/p} ds < \infty$.

At this point we roughly recall that the nonexistence of entire solutions for coercive problems is connected with the validity of condition (KO), while the failure of (KO) gives existence of entire solutions. In particular, in the latter case Theorem 1.5 of [8], relative to the Euclidean case, shows that we can expect only unbounded solutions or equivalently *large solutions*. We are now in a position to extend and to generalize in several directions the core of Corollary 1.4 of [10], without requiring any monotonicity on ℓ.

Theorem 1.1. *Let* $f(0) = 0$ *and* $\ell(0) > 0$ *in* ($\mathscr{H}$). *Then* (1.1) *admits a nonnegative local radial stationary* C^1 *solution. If furthermore* f *is nondecreasing in* $\mathbb{R}_0^+$ *and*

$$\int^{\infty} \frac{dt}{K^{-1}(F(t))} = \infty \tag{$VsKO$}$$

holds, then (1.1) *possesses a nonnegative entire large radial stationary solution* u *of class* $C^1(\mathbb{H}^m)$. *Finally, if in addition*

$$\int_{0^+} \frac{dt}{K^{-1}(F(t))} = \infty \tag{1.5}$$

is valid, then $u > 0$ *in* $\mathbb{H}^m$.

The requests of Theorem 1.1 are fairly natural and general. Theorem 1.1 can be applied not only in the p-Laplacian case, $\varphi(s) = s^{p-1},\ p > 1$, but also in the generalized mean curvature case, $\varphi(s) = s(1+s^2)^{(p-2)/2},\ p \in (1, 2)$. For other elliptic operators we refer to [25,2].

The next result concerns uniqueness of radial stationary solutions of (1.1), as in Theorem 1.1 we do not require any monotonicity assumption on ℓ in $\mathbb{R}_0^+$.

Theorem 1.2. *Assume that* f *and* ℓ *are locally Lipschitz continuous in* $\mathbb{R}_0^+$, *that* $\ell(0) > 0$ *and finally that* $\varphi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$. *Then, for each fixed* $u_0 > 0$ *Eq.* (1.1) *admits a unique radial stationary solution* u, *with* $u(O) = u_0$, *where* O *is the natural origin in* $\mathbb{H}^m$, *in the open maximal ball* B_R *of* $\mathbb{H}^m$.