Nonlinear Analysis 121 (2015) 336–369

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Nonlinear elliptic equations with variable exponent: Old and new☆

Vicenţiu D. Rădulescu *

Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania

ARTICLE INFO

Communicated by S. Carl

To Professor Enzo L. Mitidieri, with cordial wishes for his 60th birthday

MSC:
primary 35J70
secondary 35D05
35J60
58E05
74M05
76A05

Keywords:
Nonlinear elliptic operator
Variable exponent
Variational methods
Continuous spectrum

ABSTRACT

In this survey paper, by using variational methods, we are concerned with the qualitative analysis of solutions to nonlinear elliptic problems of the type

$$\begin{cases} -\mathrm{div}\, A(x, \nabla u) = \lambda |u|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded or an exterior domain of $\mathbb{R}^N$ and q is a continuous positive function. The results presented in this paper extend several contributions concerning the Lane–Emden equation and we focus on new phenomena which are due to the presence of variable exponents.

1. Introduction

One of the reasons of the huge development of the theory of classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ (where $1 \leqslant p \leqslant \infty$) is the description of many phenomena arising in applied sciences. For instance, many materials can be modeled with sufficient accuracy using the function spaces L^p and $W^{1,p}$, where p is a fixed constant. For some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as "smart fluids"), this approach is not adequate, but rather the exponent p should be able to vary. This leads us to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real-valued function. Variable exponent Lebesgue spaces appeared in the literature for the first time already in a paper by Orlicz [38]. In the 1950s this study was carried on by Nakano [37], who made the first systematic study of spaces with variable exponent (called modular spaces). Nakano mentioned explicitly variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano [37, p. 284]. Later, the Polish mathematicians investigated the modular function spaces, see Musielak [35]. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the work of Tsenov [45], Sharapudinov [43] and Zhikov [47]. We refer to the monograph by Diening, Harjulehto, Hästö, and Ruzicka [10] for a comprehensive introduction to

☆ This work was supported by Grant CNCS PCE-47/2011.
* Correspondence to: Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia.
E-mail address: vicentiu.radulescu@math.cnrs.fr.

http://dx.doi.org/10.1016/j.na.2014.11.007
0362-546X/

the theory of function spaces with variable exponents and various applications. We also point out the multiple contributions of the Finnish "Research group on variable exponent spaces and image processing" [22], whose main purpose is to study nonlinear potential theory in variable exponent Sobolev spaces.

This paper is motivated by phenomena which are described by nonlinear boundary value problems of the type

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = f(x, u), & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded or an exterior domain with smooth boundary.

The interest in studying such problems consists in the presence of the Laplace-type operator with variable exponent $\mathrm{div}(a(x, \nabla u))$. A basic example is the $p(x)$-Laplace operator, which is defined by

$$\Delta_{p(x)} u = \mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u).$$

The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. In 1920, E. Bingham was surprised that some paints do not run, like honey. He studied such a behavior and described a strange phenomenon. There are fluids that flow then stop spontaneously (Bingham fluids). Within them, the forces that create flow reach a first threshold. As this threshold is not reached, the fluid flows without deforms as a solid. Invented in the 17th century, the "Flemish medium" makes painting oil thixotropic: it fluidies under pressure of the brush, but freezes as soon as you leave the rest. While the exact composition of the medium Flemish remains unknown, it is known that the bonds form gradually between its components, which is why the picture freezes in a few minutes. Thanks to this wonderful medium, Rubens has painted *La Kermesse* in 24 h.

Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery on electrorheological fluids is due to Willis Winslow, who obtained a US patent on the effect in 1947 and wrote an article published in 1949, see [46]. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium Polymethacrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics, we refer to consult Halsey [19]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modeling and the application of variable exponent spaces to these fluids we refer to Acerbi and Mingione [1], Ruzicka [42], Chen, Levine, and Rao [9], Harjulehto, Hästö, Latvala, and Toivanen [20], Molica Bisci and Repovš [34], et al. We also point out the pioneering contributions of Pucci et al. [4,5,3] in the study of Kirchhoff-type problems, including nonlocal problems with variable exponent like

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x, u).$$

We give in what follows two relevant examples that justify the mathematical study of models involving variable exponents.

Example 1 (*Image Restoration Chen, Levine, Rao [9]*)**.** In image restoration, we consider an input I that corresponds to shades of gray in a domain $\Omega \subset \mathbb{R}^2$. We assume that I is made up of the true image corrupted by noise. Suppose that the noise is additive, that is, $I = T + \eta$ where T is the true image and η is a random variable with zero mean. Thus, the effect of the noise can be eliminated by smoothing the input, since this will cause the effect of the zero-mean random variables at nearby locations to cancel. Smoothing corresponds to minimizing the energy

$$\mathcal{E}_1(u) = \int_\Omega (|\nabla u(x)|^2 + |u(x) - I(x)|^2)dx.$$

Unfortunately, smoothing will also destroy the small details from the image, so this procedure is not useful A better approach is *total variation smoothing*. Since an edge in the image gives rise to a very large gradient, the level sets around the edge are very distinct, so this method does a good job of preserving edges. Total variation smoothing corresponds to minimizing the energy

$$\mathcal{E}_2(u) = \int_\Omega (|\nabla u(x)| + |u(x) - I(x)|^2)dx.$$

Unfortunately, total variation smoothing not only preserves edges, but also creates edges where there were none in the original image. This is the *staircase effect*.

Looking at $\mathcal{E}_1$ and $\mathcal{E}_2$, Chen, Levine and Rao suggest that an appropriate energy is

$$\mathcal{E}(u) = \int_\Omega (|\nabla u(x)|^{p(x)} + |u(x) - I(x)|^2)dx,$$

where $1 \leqslant p \leqslant 2$.