Nonlinear Analysis 121 (2015) 370–381

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Some Liouville theorems for the fractional Laplacian

Wenxiong Chen [a,b,*], Lorenzo D'Ambrosio [c], Yan Li [b]

[a] College of Mathematics and Information Science, Henan Normal University, China
[b] Department of Mathematical Sciences, Yeshiva University, New York, NY, 10033, USA
[c] Dipartimento di Matematica, Università degli Studi di Bari, via Orabona, 4 - 70125 Bari, Italy

ARTICLE INFO

Communicated by S. Carl

Dedicated to Professor Enzo Mitidieri on the occasion of his 60th birthday

Keywords:
The fractional Laplacian
α-harmonic functions
Liouville theorem
Poisson representations
Fourier analysis

ABSTRACT

In this paper, we prove the following result. Let α be any real number between 0 and 2. Assume that u is a solution of

$$\begin{cases} (-\triangle)^{\alpha/2} u(x) = 0, & x \in \mathbb{R}^n, \\ \displaystyle\lim_{|x|\to\infty} \frac{u(x)}{|x|^{\gamma}} \geq 0, & \end{cases}$$

for some $0 \leq \gamma \leq 1$ and $\gamma < \alpha$. Then u must be constant throughout $\mathbb{R}^n$.

This is a Liouville Theorem for α-harmonic functions under a much weaker condition.

For this theorem we have two different proofs by using two different methods: One is a direct approach using potential theory. The other is by Fourier analysis as a corollary of the fact that the only α-harmonic functions are affine.

1. Introduction

The well-known classical Liouville's Theorem states that

Any harmonic function bounded below in all of R^n is constant.

One of its important applications is the proof of the Fundamental Theorem of Algebra. It is also a key ingredient in deriving a priori estimates for solutions in PDE analysis.

This Liouville Theorem has been generalized to the fractional Laplacian by Bogdan, Kulczycki, and Nowak [2]:

Proposition 1.1. *Let $0 < \alpha < 2$ and $n \geq 2$. Assume that u is a solution of*

$$\begin{cases} (-\triangle)^{\alpha/2} u(x) = 0, & x \in \mathbb{R}^n, \\ u(x) \geq 0, & x \in \mathbb{R}^n. \end{cases}$$

Then u must be constant.

The same result has been proved by Zhuo, Chen, Cui, and Yuan [25] using a completely different method; and then interesting applications of this Liouville theorem to integral representations of solutions for nonlinear equations and systems involving the fractional Laplacian were investigated in the same article.

* Corresponding author at: Department of Mathematical Sciences, Yeshiva University, New York, NY, 10033, USA.
E-mail addresses: wchen@yu.edu (W. Chen), dambros@dm.uniba.it (L. D'Ambrosio), yali3@mail.yu.edu (Y. Li).

http://dx.doi.org/10.1016/j.na.2014.11.003
0362-546X/

The above proposition has also been proved but not explicitly stated in [14] for α-harmonic functions in the average sense (see also Section 4 for the definition). Indeed it can be deduced from [14, Theorem 1.30].

In [1], Axler, Bourdon, and Ramey replaced the condition "bounded below" in the classical Liouville Theorem by a much weaker one:

$$\varliminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0.$$

Then Enzo Mitidieri [18] conjectured that a similar result should hold for the fractional Laplacian. The main purpose of this paper is to prove this conjecture.

The fractional Laplacian in R^n is a nonlocal pseudo-differential operator, taking the form

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \lim_{\epsilon\to 0} \int_{R^n\setminus B_\epsilon(x)} \frac{u(x)-u(z)}{|x-z|^{n+\alpha}} dz, \tag{1}$$

where α is any real number between 0 and 2. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in R^n. In this space, it can also be equivalently defined in terms of the Fourier transform

$$\mathcal{F}((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi),$$

where $\mathcal{F}(u)$ is the Fourier transform of u. One can extend this operator to a wider space of distributions.

Let

$$L_\alpha = \left\{ u : R^n \to R \mid \int_{R^n} \frac{|u(x)|}{1+|x|^{n+\alpha}}\, dx < \infty \right\}.$$

Then in this space, one can defined $(-\Delta)^{\alpha/2}u$ as a distribution by

$$\langle(-\Delta)^{\alpha/2}u(x), \phi\rangle = \int_{R^n} u(x)(-\Delta)^{\alpha/2}\phi(x)dx, \quad \forall \phi \in C_0^\infty(R^n).$$

The operator can be also defined by considering the following problem in $\mathbb{R}^n \times [0, +\infty)$:

$$f(x,0) = u(x) \quad \text{on } \mathbb{R}^n, \qquad \triangle_x f + \frac{1-\alpha}{y} f_y + f_{yy} = 0$$

and defining

$$C(-\Delta)^{\alpha/2}u(x) := -\lim_{y\to 0} y^{1-\alpha} f_y(x,y),$$

where $C = C(n,\alpha)$ is a suitable positive constant. See [5] for more details.

Throughout this paper, we will consider the fractional Laplacian defined in (1). We say that u is α-harmonic if $u \in L_\alpha$ and satisfies

$$(-\Delta)^{\alpha/2}u = 0$$

in the sense of distributions:

$$\int_{R^n} u(x)(-\Delta)^{\alpha/2}\phi(x)dx = 0 \quad \forall \phi \in C_0^\infty(R^n).$$

Our main objective is to prove the following result.

Theorem 1.2. *Let $0 < \alpha < 2$. Assume that $u \in L_\alpha$ is α-harmonic and*

$$\varliminf_{|x|\to\infty} \frac{u(x)}{|\,x\,|^\gamma} \geq 0, \tag{2}$$

for some $0 \leq \gamma \leq 1$ and $\gamma < \alpha$.

Then

$$u(x) \equiv C, \quad x \in \mathbb{R}^n.$$

We prove the above result by two different methods. The first one is given in Section 2 and it is a direct proof based on classical potential theory.

The second proof is a direct consequence of the following

Theorem 1.3. *Let $0 < \alpha < 2$, and $u \in L_\alpha$ be α-harmonic. Then u is affine. In particular, if $\alpha \leq 1$, then u is constant.*