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A priori estimates and bifurcation of solutions for an elliptic equation with semidefinite critical growth in the gradient

Philippe Souplet

Université Paris 13, Sorbonne Paris Cité, CNRS UMR 7539, Laboratoire Analyse Géométrie et Applications, 93430 Villetaneuse, France

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ABSTRACT

We study nonnegative solutions of the boundary value problem

$$-\Delta u = \lambda c(x)u + \mu(x) \mid \nabla u \mid^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \qquad (P_\lambda)$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, $\mu, c \in L^\infty(\Omega)$, $h \in L^r(\Omega)$ for some $r > n/2$ and $\mu, c, h \gneqq 0$. Our main motivation is to study the "semidefinite" case. Namely, unlike in previous work on the subject, we do not assume μ to be uniformly positive in Ω, nor even positive everywhere.

In space dimensions up to $n = 5$, we establish uniform a priori estimates for weak solutions of (P_λ) when $\lambda > 0$ is bounded away from 0. This is proved under the assumption that the supports of μ and c intersect, a condition that we show to be actually necessary, and in some cases we further assume that μ is uniformly positive on the support of c and/or some other conditions.

As a consequence of our a priori estimates, assuming that (P_0) has a solution, we deduce the existence of a continuum $\mathcal{C}$ of solutions, such that the projection of $\mathcal{C}$ onto the λ-axis is an interval of the form $[0, a]$ for some $a > 0$ and that the continuum $\mathcal{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$. In particular, for each $\lambda > 0$ small enough, problem (P_λ) has at least two distinct solutions.

1. Introduction and main results

In this article, we consider the following Dirichlet problem:

$$\begin{cases} -\Delta u = \mu(x)|\nabla u|^2 + \lambda c(x)u + h(x), \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \qquad (P_\lambda)$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^2, μ, c, h are given functions, whose regularity will be specified below, and λ is a real parameter. By a solution, we mean a weak solution in the sense of the usual integral formulation, with test-functions in $H_0^1(\Omega) \cap L^\infty(\Omega)$. It is known (see [5]) that, under assumption (1.3), any solution u of (P_λ) is Hölder continuous in $\overline{\Omega}$.

Elliptic equations with a gradient dependence up to the critical (quadratic) growth were studied by Boccardo, Murat and Puel in the 80's and a large literature on the subject has appeared since then. Many results are known in the case $\lambda = 0$ (see e.g. [3,20,2,14,18,15,13]) or $\lambda < 0$ (see e.g. [8,9,4,5]). We shall here consider the case $\lambda > 0$ and will be concerned with questions of a priori estimates, existence, multiplicity and bifurcation of solutions.

E-mail address: souplet@math.univ-paris13.fr.

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Denote the nonnegative solution set

$$\Sigma = \left\{(\lambda, u) \in [0,\infty) \times C(\overline{\Omega});\ u \geq 0 \text{ and } u \text{ solves } (P_\lambda)\right\}.$$

Interesting properties of the set Σ were recently established in [5] under the assumptions (with $n \geq 3$):

$$\mu \in L^\infty(\Omega), \quad c, h \in L^r(\Omega) \text{ for some } r > n/2,\ c, h \gneqq 0, \tag{1.1}$$

$$\mu(x) \geq \mu_0 > 0. \tag{1.2}$$

For every $\varepsilon > 0$, it is shown that nonnegative solutions of (P_λ) with $\lambda \geq \varepsilon$ satisfy a uniform a priori estimate. Next assume in addition that (P_0) has a (necessarily nonnegative) solution (see Remark 1.1 for known sufficient conditions). Then it is shown in [5] that there exists a continuum $\mathcal{C} \subset \Sigma$, such that the projection of $\mathcal{C}$ onto the λ-axis is an interval of the form $[0, a]$ for some $a > 0$ and that the continuum $\mathcal{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$. In particular, for each $\lambda > 0$ small enough, $\mathcal{C}$ contains at least two distinct solutions of (P_λ).

We note that, when $\mu(x) = \mu$ is a positive constant, multiplicity results (actually up to an explicit value $\lambda_0 > 0$ of λ) have been obtained before in [19], using the transformation $v = e^{\mu u}$ (see also [1]). We stress that the multiplicity results in [5] allow nonconstant functions $\mu(x)$, in which case such a transformation is not available. However, the uniform positivity of the function $\mu(x)$, i.e. (1.2), is still needed in [5].

Our main goal here is to establish similar results as in [5] for semidefinite functions $\mu(x)$, namely to allow just $\mu \geq 0$, $\mu \not\equiv 0$ (possibly at the expense of additional assumptions on c). The main difficulty here is to establish a priori estimates without assuming (1.2). Indeed, this assumption seems necessary in [5], in order to apply the method of Brezis and Turner [11] based on Hardy–Sobolev inequalities. Therefore, we need some new ideas (see after the statement of the results, for a brief description of the main arguments of our proofs).

We assume:

$$\mu, c \in L^\infty(\Omega), \quad h \in L^r(\Omega) \text{ for some } r > \max(1, n/2),\ \mu, c, h >\gneqq 0. \tag{1.3}$$

Also, we shall make the following essential assumption of *intersecting supports* for μ and c:

$$\mu, c \geq \eta \quad \text{on } B(x_0, \rho) \subset \Omega \text{ for some } \rho,\ \eta > 0. \tag{1.4}$$

In low dimensions $n \leq 2$, this turns out to be sufficient to guarantee a priori estimates.

Theorem 1. *Let $n \leq 2$ and assume (1.3), (1.4). Then for any $\Lambda_1 > 0$ there exists a constant $M > 0$ such that, for each $\lambda \geq \Lambda_1$, any nonnegative solution of (P_λ) satisfies $\|u\|_\infty \leq M$.*

We shall see right away that the assumption (1.3) of intersecting supports for μ and c is also essentially necessary.

Theorem 2. *Consider problem (P_λ) with*

$$n = 1, \qquad \Omega = (0, 3), \qquad \mu(x) = \chi_{(1,2)}, \qquad c(x) = \chi_{(2,3)}, \qquad h = 0.$$

There exists a sequence $\lambda = \lambda_j \to \pi^2/4$ and a sequence $u_j \in H^2(\Omega) \cap H^1_0(\Omega)$ of solutions of (P_λ) such that $\|u_j\|_\infty \to \infty$ as $j \to \infty$.

We now turn to the higher dimensional range $3 \leq n \leq 5$. In this case, beside (1.4), we need additional assumptions to guarantee a priori estimates. In our next result, we shall assume, roughly speaking, that μ *is positively bounded below on the support of* c. However, we do not know presently whether this assumption is technical or not (nor the assumptions in Theorem 4). In particular, we do not know if the a priori estimates are still true in dimensions $n \geq 6$. The restriction $n \leq 5$ comes from the fact that our method (see at the end of this section) requires to estimate the function $c(x)u$ in L^p for some $p > n/2$, whereas the L^p estimates that we are able to derive are limited to $p \leq 2n/(n-2)$, due to Sobolev embeddings or to even more stringent functional inequalities (note that $2n/(n-2)$ and $n/2$ precisely coincide for $n = 6$).

Theorem 3. *Let $3 \leq n \leq 5$ and let (1.3), (1.4) be satisfied. Assume that there exists a C^2 domain $\omega \subset \Omega$ and a constant $\mu_0 > 0$, such that*

$$\mu \geq \mu_0 \quad \text{on } \omega \text{ and } \mathrm{Supp}(c) \subset \overline{\omega}. \tag{1.5}$$

If $n = 5$, assume in addition that

$$c(x) \leq C_1[\mathrm{dist}(x, \partial\omega)]^\sigma, \quad x \in \omega, \text{ for some } C_1, \sigma > 0. \tag{1.6}$$

Then for any $\Lambda_1 > 0$ there exists a constant $M > 0$ such that, for each $\lambda \geq \Lambda_1$, any nonnegative solution of (P_λ) satisfies $\|u\|_\infty \leq M$.

As usual, Supp is here understood in the sense of essential support. As a special case of Theorem 3, we see that the a priori estimate holds for instance if $3 \leq n \leq 5$, c is compactly supported and $\mu \geq \mu_0 > 0$ on a neighborhood of the support of c.

We now give additional results in the case of dimension $n = 3$, which is rather special. Indeed, without assuming (1.5), we can then obtain a priori estimates under various, relatively mild, assumptions either on μ or c.

Theorem 4. *Let $n = 3$ and (1.3), (1.4) be satisfied. Assume in addition that either*

$$c(x) \leq C_1[\mathrm{dist}(x, \partial\Omega)]^\sigma, \quad x \in \Omega, \text{ for some } C_1, \sigma > 0, \tag{1.7}$$