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Multiple perturbations of a singular eigenvalue problem

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ABSTRACT

We study the perturbation by a critical term and a $(p-1)$-superlinear subcritical nonlinearity of a quasilinear elliptic equation containing a singular potential. By means of variational arguments and a version of the concentration-compactness principle in the singular case, we prove the existence of solutions for positive values of the parameter under the principal eigenvalue of the associated singular eigenvalue problem.

1. Introduction

Let $\Omega \subseteq \mathbb{R}$ be an *arbitrary* open set, $1 < p < N$, and let $\mathcal{D}_0^{1,p}(\Omega)$ denote the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|u\| := (\int_\Omega |\nabla u|^p dx)^{1/p}$. Let $V \in L^1_{loc}(\Omega)$ be a function which may have strong singularities and an indefinite sign.

Smets was interested in [1] in finding nontrivial weak solutions for the following nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda V(x)|u|^{p-2}u & \text{in } \Omega \\ u \in \mathcal{D}_0^{1,p}(\Omega). \end{cases} \tag{1}$$

Problems of this type are in relationship with the study of the standing waves in the anisotropic Schrödinger or Klein–Gordon equations, cf. Reed and Simon [2], Strauss [3], and Wang [4]. Eq. (1) is also considered a model for several physical phenomena related to the equilibrium of anisotropic media that possibly are somewhere *perfect insulators* or *perfect conductors*, see Dautray and Lions [5, p. 79]. We point out that degenerate or singular problems have been intensively studied starting with the pioneering paper by Murthy and Stampacchia [6].

Problem (1) is in relationship with several papers dealing with nonlinear anisotropic eigenvalue problems, see Brown and Tertikas [7], Rozenblioum and Solomyak [8]. Szulkin and Willem generalize in [9] several earlier results concerning the

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existence of an infinite sequence of eigenvalues. The main hypothesis on the potential V in [9] is the following:

$$\begin{cases} V \in L^1_{loc}(\Omega), \quad V^+ = V_1 + V_2 \neq 0, \quad V_1 \in L^{N/p}(\Omega), \\ \text{for every } y \in \overline{\Omega}, \qquad \lim_{x\to y, x\in\Omega} |x-y|^p V_2(x) = 0 \quad \text{and} \\ \lim_{x\to\infty, x\in\Omega} |x|^p V_2(x) = 0. \end{cases} \tag{2}$$

Under assumption (2), the mapping $\mathcal{D}_0^{1,p}(\Omega) \ni u \longmapsto \int_\Omega V^+|u|^p dx$ is weakly continuous, so the problem is not affected by a lack of compactness. In [1] the case of indefinite potential functions V is studied for which no *a priori* compactness is assumed. The corresponding hypotheses extend condition (2), nonetheless they are not directly linked to punctual growths of V. Due to the presence of a singular potential, the classical methods cannot be applied directly, so the existence can become a delicate matter.

Consider the minimization problem

$$S_V := \inf\left\{\int_\Omega |\nabla u|^p dx;\ u \in \mathcal{D}_0^{1,p}(\Omega),\ \int_\Omega V(x)|u|^p dx = 1\right\}. \tag{3}$$

As established in [1] with standard constrained minimization arguments, minimizers of problem (3) correspond to weak solutions of (1), with λ appearing as a Lagrange multiplier (that is, $\lambda = S_V$). Such a parameter λ is called the *principal eigenvalue* for problem (1).

In order to have $S_V \neq 0$ and well defined, we assume that $V = V^+ - V_-$, $V^+ \neq 0$, and that there exists $c > 0$ such that for all $u \in \mathcal{D}_0^{1,p}(\Omega)$,

$$c\int_\Omega V^+|u|^p dx \le \int_\Omega |\nabla u|^p dx. \tag{4}$$

By Hardy's inequality it follows that potentials with point singularities and decay at infinity both at most as $O(|x|^{-p})$ satisfy hypothesis (4).

Since Ω is not necessarily bounded and V can have singularities, it is not clear that the infimum in problem (3) is achieved without imposing additional conditions that allow the analysis of minimizing sequences. For all $x \in \overline{\Omega}$ and $r > 0$, we denote by $B_r(x)$ the open ball centered at x and of radius r and by B_r the closed ball centered at the origin (we can assume without any loss of generality that $0 \in \Omega$). We introduce the following quantities:

$$S_{r,V} := \inf\left\{\int_\Omega |\nabla u|^p dx;\ u \in \mathcal{D}(\Omega \setminus B_r),\ \int_\Omega V^+(x)|u|^p dx = 1\right\};$$

$$S_{\infty,V} := \sup_{r>0} S_{r,V} = \lim_{r\to\infty} S_{r,V};$$

$$S^x_{r,V} := \inf\left\{\int_\Omega |\nabla u|^p dx;\ u \in \mathcal{D}(\Omega \cap B_r(x)),\ \int_\Omega V^+(x)|u|^p dx = 1\right\};$$

$$S^x_V := \sup_{r>0} S^x_{r,V} = \lim_{r\to 0} S^x_{r,V};$$

$$S_{*,V} := \inf_{x\in\overline{\Omega}} S^x_V;$$

$$\Sigma_V := \{x \in \overline{\Omega};\ S^x_V < \infty\}.$$

Applying Hardy's inequality

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \le \left(\frac{N}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx,$$

we observe that under assumption (2) introduced in [9], we have $S_{\infty,V} = S_{*,V} = +\infty$. As argued in [1, p. 475], the condition $S_{\infty,V} = S_{*,V} = +\infty$ is equivalent to the weak continuity of the mapping $u \longmapsto \int_\Omega V^+(x)|u|^p dx$.

We make the following hypothesis:

$$\text{the closure of } \Sigma_V \text{ is at most countable.} \tag{5}$$

In particular, condition (5) excludes the presence of strong *spikes* on a dense subset of Ω.

For $V \in L^1_{loc}(\Omega)$ satisfying assumptions (4) and (5), Smets proved in [1] that the singular eigenvalue problem (1) admits a principal eigenvalue, provided that $S_V < S_{\infty,V}$ and $S_V < S_{*,V}$. This result extends and simplifies the work of Tertikas [10], which deals with the positive linear case for $\Omega = \mathbb{R}^N$. We point out (see [1, p. 472]) that the condition $p < N$ is necessary only if Ω is unbounded, otherwise one can work in the standard Sobolev space $W_0^{1,p}(\Omega)$.

We are interested in studying what happens if problem (1) is affected by certain perturbations. This is needed in several applications and the idea of using perturbation methods in the treatment of nonlinear boundary value problems was introduced by Struwe [11]. Existence results for nonautonomous perturbations of critical singular elliptic boundary value