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Weak anchoring for a two-dimensional liquid crystal

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ABSTRACT

We study the weak anchoring condition for nematic liquid crystals in the context of the Landau–De Gennes model. We restrict our attention to two dimensional samples and to nematic director fields lying in the plane, for which the Landau–De Gennes energy reduces to the Ginzburg–Landau functional, and the weak anchoring condition is realized via a penalized boundary term in the energy. We study the singular limit as the length scale parameter $\varepsilon \to 0$, assuming the weak anchoring parameter $\lambda = \lambda(\varepsilon) \to \infty$ at a prescribed rate. We also consider a specific example of a bulk nematic liquid crystal with an included oil droplet and derive a precise description of the defect locations for this situation, for $\lambda(\varepsilon) = K\varepsilon^{-\alpha}$ with $\alpha \in (0, 1]$. We show that defects lie on the weak anchoring boundary for $\alpha \in (0, \frac{1}{2})$, or for $\alpha = \frac{1}{2}$ and K small, but they occur inside the bulk domain Ω for $\alpha > \frac{1}{2}$ or $\alpha = \frac{1}{2}$ with K large.

1. Introduction

In this paper we examine the weak anchoring condition for nematic liquid crystals in the context of the Landau–De Gennes model. Weak anchoring refers to the imposition of boundary behavior by means of energy penalization, rather than via a nonhomogeneous Dirichlet condition (which is referred to as "strong anchoring"). We restrict our attention to two-dimensional samples and to nematic director fields lying in the plane. With this dimensional restriction, the Landau–De Gennes energy reduces to the familiar Ginzburg–Landau energy, for a complex valued order parameter u which is mapped to the Q-tensor in the Landau–De Gennes theory, and the weak coupling condition is expressed as a boundary penalization term added to the Ginzburg–Landau energy. We study the singular limit as the length scale parameter $\varepsilon \to 0$, assuming the weak anchoring penalization strength $\lambda = \lambda(\varepsilon) \to \infty$ at a prescribed rate. We also consider a specific example of a bulk nematic liquid crystal with an included oil droplet [1], and derive a precise description of the defect locations for this situation, depending on the relative strength of the weak anchoring parameter $\lambda(\varepsilon)$. Although the Ginzburg–Landau functional represents a highly simplified model for nematic liquid crystals, we expect that it nevertheless captures the salient information concerning the formation of singularities under the weak anchoring condition.

We first describe our results in the context of the Ginzburg–Landau model with boundary penalization; the description of the Landau–De Gennes model and the physical droplet setting, together with the reduction to the Ginzburg–Landau energy, will be explained afterwards. In particular, the solution to the droplet problem is stated in Theorem 1.2. Let

$$\lambda = \lambda(\varepsilon) = K\varepsilon^{-\alpha}$$

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for $\alpha \in (0, 1], K > 0$ constant. We impose the weak anchoring condition on a connected component Γ of $\partial\Omega$ via a boundary term in the energy. Let $g : \Gamma \to S^1$ be a C^2 smooth map, and define

$$E_\varepsilon(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(|u|^2 - 1\right)^2\right) dx + \frac{\lambda}{2}\int_\Gamma |u - g|^2\, dS.$$

A critical point of $E_\varepsilon(u)$ in $H^1(\Omega; \mathbb{C})$ solves

$$\left.\begin{aligned} &-\Delta u + \frac{1}{\varepsilon^2}(|u|^2 - 1)u = 0, \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} + \lambda(u - g) = 0, \quad \text{on } \Gamma. \end{aligned}\right\} \tag{1.1}$$

We consider three different geometries, each with some physical motivation.
Problem I: $\Omega \subset \mathbb{R}^2$ is simply connected and with smooth C^2 boundary $\partial\Omega = \Gamma$. In this case, the appropriate space is $\mathbb{H}_I := H^1(\Omega; \mathbb{C})$, and (1.1) gives the Euler–Lagrange equations corresponding to this variational problem.
Problem II: $\Omega = \Omega_1 \setminus \overline{\Omega_0}$ is a topological annulus, with C^2 smooth boundary in two components, $\Gamma = \partial\Omega_0$ the interior boundary, and $\partial\Omega_1$ the exterior. We impose weak anchoring via $g : \Gamma \to S^1$ on the interior boundary, and a constant Dirichlet condition on the exterior, so the Euler–Lagrange equations are (1.1) with the additional condition,

$$u = 1, \quad \text{on } \partial\Omega_1. \tag{1.2}$$

The appropriate space is

$$\mathbb{H}_{\text{II}} := \{u \in H^1(\Omega; \mathbb{C}) : u = 1 \text{ on } \partial\Omega_1\}.$$

The choice of a constant as a Dirichlet (strong anchoring) boundary condition is motivated by the physical model of a droplet Ω_0 included in a bulk nematic (described below); mathematically, the problem may be posed with any S^1-valued map imposed on the outer boundary $\partial\Omega_1$.
Problem III: $\Omega = \mathbb{R}^2 \setminus \Omega_0$ is an exterior domain, with boundary $\Gamma = \partial\Omega_0$. We impose a weak anchoring condition on Γ via the C^2 map $g : \Gamma \to S^1 \subset \mathbb{C}$, and assume that there exists a constant $\phi_0 \in (-\pi, \pi]$ for which

$$u(x) \to e^{i\phi_0} \quad \text{as } |x| \to \infty. \tag{1.3}$$

We minimize E_ε in the space

$$\mathbb{H}_{\text{III}} := \{u \in H^1_{\text{loc}}(\Omega; \mathbb{C}) : \exists\phi_0 \in \mathbb{R} \text{ such that } u \to e^{i\phi_0} \text{ as } |x| \to \infty\},$$

and minimizers satisfy the Euler–Lagrange equations (1.1) in the unbounded domain Ω, with asymptotic condition (1.3). As in Problem II, the choice of a constant at infinity is motivated by the droplet problem posed in [1].

The space $\mathbb{H}_{\text{III}}$ is problematic, as the Dirichlet energy does not control the phase of u as $|x| \to \infty$, and in fact the existence of minimizers for fixed $\varepsilon > 0$ is not immediate. Indeed, unlike the Dirichlet problems I and II, we may not specify a limiting constant as $|x| \to \infty$; the asymptotic phase ϕ_0 is an unknown quantity in the problem, determined by the choice of Ω_0 and g. In the application to nematic liquid crystals, $\Omega_0 = D_1(0)$ a disk, and $g = e^{iD\theta}$ is symmetric, and in this case we may in fact conclude that the energy minimizers satisfy $u(x) \to 1$ as $|x| \to \infty$ (see Theorem 2.1).

Our aim in this paper is to study the minimizers of E_ε as $\varepsilon \to 0$, for each problem I, II, III, and determine how the location of the vortices is affected by the weak anchoring strength $\lambda = \lambda(\varepsilon) = K\varepsilon^{-\alpha}$. In particular, we observe that $\alpha = \frac{1}{2}$ is the critical value for the weak anchoring strength, with vortices lying on the boundary component Γ for $\alpha < \frac{1}{2}$ and inside Ω for $\alpha > \frac{1}{2}$. Here is our main result for Problems I, II, and III:

Theorem 1.1. *Let $g : \Gamma \to S^1$ be a given C^2 function with degree $\mathcal{D} \in \mathbb{N}$. Let u_ε be minimizers of E_ε in one of the spaces $\mathbb{H}_i$, $i =$ I, II, III. For any sequence of $\varepsilon \to 0$ there is a subsequence $\varepsilon_n \to 0$ and $\mathcal{D}$ points $\{p_1, \ldots, p_{\mathcal{D}}\}$ in $\Omega \cup \Gamma$ such that*

$$u_{\varepsilon_n} \to u_* \quad \text{in } C^{1,\mu}_{\text{loc}}(\overline{\Omega} \setminus \{p_1, \ldots, p_{\mathcal{D}}\}),$$

for $0 < \mu < 1$, with $u_* : \Omega \setminus \{p_1, \ldots, p_{\mathcal{D}}\} \to S^1$ a harmonic map. Moreover,

(a) $u_* = g$ *on* $\Gamma \setminus \{p_1, \ldots, p_{\mathcal{D}}\}$.
(b) *For each* $i = 1, \ldots, \mathcal{D}$, $\deg(u_*; p_i) = 1$ *in problem I, and* $\deg(u_*; p_i) = -1$ *in problems II and III.*
(c) *If* $0 < \alpha < \frac{1}{2}$, *each* $p_i \in \Gamma$; *if* $\frac{1}{2} < \alpha \le 1$, *then* $p_i \in \Omega$ *for all* $i = 1, \ldots, \mathcal{D}$.
(d) *If* $\alpha = \frac{1}{2}$, *there exist* $K_0 < K_1 \in \mathbb{R}$ *such that the vortices lie on* Γ *for* $K < K_0$ *and they lie inside* Ω *for* $K > K_1$.
(e) *There are Renormalized Energy functions* $W_\Omega : \Omega^{\mathcal{D}} \to \mathbb{R}$ *and* $W_\Gamma : \Gamma^{\mathcal{D}} \to \mathbb{R}$ *such that if* $(p_1, \ldots, p_{\mathcal{D}})$ *lie on* Γ, *they minimize* W_Γ, *and if they lie inside* Ω *they minimize* W_Ω.