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Free boundaries touching the boundary of the domain for some reaction–diffusion problems

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ABSTRACT

We give conditions on the behaviour of the trace datum near the boundary of its support in order to know whether the free boundary given by the boundary of the support of the solution of suitable elliptic or parabolic semilinear problem is connected or not with the boundary of the support of the boundary datum.

1. Introduction

We study the way the free boundary of solutions to some partial differential equations behaves depending on the trace of the solutions. The free boundary problems we consider are of two different types:

(i) *Elliptic reaction–diffusion type problems*, as

$$\begin{cases} -Lu + \lambda u^q = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{1}$$

under the fundamental assumption

$$q \in (0, 1), \tag{2}$$

which guarantees the formation of the free boundary (at least for $\lambda > 0$ large enough, if Ω is bounded, or for any $\lambda > 0$, if Ω is unbounded). Such problem arises, for instance, in Chemical Engineering when a catalytic chemical reactor occupying a domain Ω has a reactant feed channel (entrance boundary) which is represented by the part $\Gamma_+ \subset \partial\Omega$, where the reactant concentration is $h(x) > 0$ and the rest of walls of the chemical reactant are isolated in such a way that, if we denote by $\Gamma_0 := \partial\Omega \setminus \Gamma_+$, then $h(x) = 0$ on Γ_0. Here we assume that there is no exit boundary (see Fig. 1). The exponent q is called the order of the reaction.

(ii) *The obstacle problem*

$$\begin{cases} -Lu \geq f(x), \quad u \geq 0 \quad \text{and} \quad (-Lu - f(x))u = 0 & \text{in } \Omega, \\ u(x) = h(x) & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here the free boundary is given by the boundary of the *coincidence set* (the set of points where $u = 0$); according for instance to [17] a sufficient condition for the existence of the free boundary is that $f(x) \leq -\mu$ for some $\mu > 0$ on a large enough

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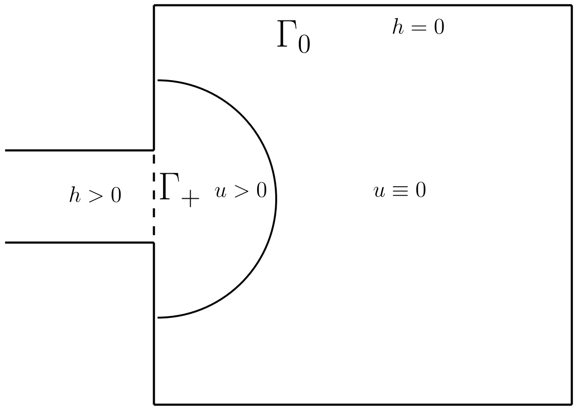

Fig. 1. Chemical reactor scheme.

open subset of Ω (see, for instance, [28] for a full treatment of the obstacle problem). Among the many frameworks in which the obstacle problem arises we could mention, for instance, the unilateral problem of the stationary shape of a membrane which is forced downwards by a constant force f, is fixed on the boundary to a height $h(x)$ and constrained to lie over the hyperplane $u = 0$. Actually, here we shall consider the special case in which (3) can be formulated in terms of

$$\begin{cases} -Lu + \lambda\beta(u) \ni \varepsilon & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{4}$$

for some constant $\varepsilon \in [0, \lambda)$, where $\beta(u)$ is the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(u) = \begin{cases} 0 & \text{for } u < 0, \\ [0, 1] & \text{for } u = 0, \\ 1 & \text{for } u > 0. \end{cases} \tag{5}$$

If u "solves" problem (4) (the rigorous definition of solution will be given later) then u is also a solution of the obstacle problem (3) with $f = -\lambda + \varepsilon$: indeed, we will see that $\varepsilon \geq 0$ and $h \geq 0$ imply that $u \geq 0$. Then, if $u > 0$, $-Lu + \lambda = \varepsilon$ which is the same as $-Lu - f = 0$. Finally, since there is uniqueness of solution for both formulations we get that the solutions must be the same.

Another interesting application of problem (4) arises also in the context of Chemical Engineering (as problem (1) with $q = 0$: see, e.g., [7]).

For some general purposes, such as the existence, uniqueness and regularity of the solutions, the domain Ω will be assumed to be an open regular set of $\mathbb{R}^N$. Nevertheless, when studying the qualitative properties of the solutions we focus on the bi-dimensional case, and we adopt as domain Ω both a bounded rectangle and the upper half plain in $\mathbb{R}^2$, i.e., $\Omega = \mathbb{R} \times [0, \infty)$. In the unbounded setting we use the following notation: $x := (x_1, x_2)$ with $x_1 \in \mathbb{R}$ and $x_2 \in [0, \infty)$. The unbounded boundary of the domain is then $\partial\Omega = \mathbb{R} \times \{0\}$ and so the boundary function h will depend only on the variable x_1.

In general, L denotes a second order elliptic operator of the form

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right) = \operatorname{div}(\mathbf{A}(\mathbf{x})\nabla u), \tag{6}$$

with $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$, such that the corresponding matrix $\mathbf{A}(\mathbf{x})$ is symmetric and positive definite. Actually, in the parts concerning the behaviour of the support and free boundary of the solutions we shall restrict to the case of constant coefficients. This restriction serves merely to simplify the calculations and does not affect the local behaviour. For what concerns the boundary datum h, we assume that

$$h \in L^\infty(\partial\Omega) \quad \text{and} \quad h \geq 0 \quad \text{on } \partial\Omega,$$

even though the existence and uniqueness results on a bounded domain hold for $h \in L^1(\partial\Omega)$ (and even for signed boundary measures).

A general exposition containing many references on both problems can be found in the monograph [17]. One can see that both problems are special cases of the wider formulation

$$\begin{cases} -Lu + \lambda\beta(u) \ni f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{7}$$