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Multiple blow-up solutions for an exponential nonlinearity with potential in $\mathbb{R}^2$

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ABSTRACT

We study the following boundary value problem

$$\begin{cases} \Delta u + \lambda a(x) u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter, the function $a(x) \geq 0$ is a smooth potential, and the exponent p satisfies $0 < p < 2$. We construct a family of solutions to problem (0.1) which blows up, as $\lambda \to 0$, at some points of Ω which stay outside the zero set of $a(x)$. We relate the number of possible blow-up points with the zero set of $a(x)$.

1. Introduction

We consider the following boundary value problem

$$\begin{cases} \Delta u + \lambda a(x) u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p < 2$. The function $a(x) \geq 0$ is smooth in Ω. This problem is the Euler–Lagrange equation for the functional

$$J^p_{a,\lambda}(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \frac{\lambda}{p} \int_\Omega a(x) e^{u^p} dx, \quad u \in H^1_0(\Omega). \tag{1.2}$$

If $a(x) \equiv 1$, problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

This problem has been studied widely in the literature when $p = 1$. The asymptotic behavior of blowing up families of solutions can be referred to [1,4,13–16]: in these works it has been established that if u_λ is an unbounded family of solutions

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to (1.3) for which $\lambda \int_\Omega e^{u_\lambda}$ remains uniformly bounded as $\lambda \to 0$, then there exists an integer K such that

$$\lambda \int_\Omega e^{u_\lambda}\, dx \to 8\pi K, \quad \text{as } \lambda \to 0.$$

Moreover there are K points $\xi_1, \dots, \xi_K$ in Ω, which are far away from the boundary of Ω and far away from each other, so that

$$\lambda e^{u_\lambda} \to \sum_{j=1}^{K} \delta_{\xi_j}$$

in the sense of measure. Furthermore, the location of the point $\xi = (\xi_1, \dots, \xi_K)$ is known to be related to the critical points of the function

$$\Phi_K(\xi) = \sum_{j=1}^{K} H(\xi_j, \xi_j) + \sum_{i\neq j}^{K} G(\xi_i, \xi_j).$$

Here $G(x, y)$ denotes Green's function for the negative Laplacian with Dirichlet boundary condition in Ω, namely

$$\begin{cases} -\Delta_x G(x, y) = \delta_y(x) & x \in \Omega; \\ G(x, y) = 0 & x \in \partial\Omega, \end{cases} \tag{1.4}$$

and $H(x, y)$ its regular part, given by

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}. \tag{1.5}$$

Concerning the reciprocal issue, several results are already known in the literature, we refer to [1,10,7]. In particular, in [7] del Pino–Kowalczyk–Musso constructed bubbling solutions to problem (1.3) when $p = 1$. They showed that: *If the domain Ω is not simply connected*, and given any integer $K \geq 1$, there exist K points $\xi_1, \dots, \xi_K$ in Ω and a family of solutions u_λ, for any λ sufficiently small, which blows up at these K points in the sense that, as $\lambda \to 0$

$$\sup_{x\in\Omega\setminus\cup_{j=1}^{K} B(\xi_j,\delta)} u_\lambda(x) \to 0, \quad \text{and for any } j = 1, \dots, K, \qquad \sup_{x\in B(\xi_j,\delta)} u_\lambda(x) \to \infty$$

for any positive fixed number δ. Furthermore,

$$\int_\Omega \lambda e^{u_\lambda} dx \to 8K\pi \quad \text{as } \lambda \to 0.$$

The location of these blow-up points $\xi_1, \dots, \xi_K$ is not arbitrary: indeed they correspond to critical points of the function Φ_K defined above.

The results have been extended in [9] for the whole range of values of exponents p with $0 < p < 2$. This result was surprising, since the scenario changes completely when $p = 2$: this situation was previously treated in [8].

In this paper, we construct bubbling solutions to Problem (1.1), with a non negative nontrivial potential. When $p = 1$, this situation was already treated in [7], under the condition that the concentration points $(\xi_1, \dots, \xi_K)$ belong to a region where the potential a is strictly positive. Our first result shows that this construction can be done for the whole range of exponents $0 < p < 2$.

Before stating our result, it is useful to introduce some notations. For an integer $K \geq 1$ and K distinct points ξ_j, $j = 1, \dots, K$, in Ω, separated uniformly from each other and from the boundary $\partial\Omega$, write $\xi = (\xi_1, \dots, \xi_K)$, let us define the following functional

$$\Phi^p_{a,K}(\xi) = \sum_{j=1}^{K} H(\xi_j, \xi_j) + \sum_{i\neq j}^{K} G(\xi_i, \xi_j) + \frac{2-p}{4p\pi} \sum_{j=1}^{K} \log a(\xi_j). \tag{1.6}$$

Definition 1.1. We say that ξ is a C^0-stable critical point of $\varphi : \mathcal{M} \to \mathbb{R}$ if for any sequence of functions $\varphi_n : \mathcal{M} \to \mathbb{R}$ such that $\varphi_n \to \varphi$ uniformly on compact sets of $\mathcal{M}$, φ_n has a critical point ξ^n such that $\varphi_n(\xi^n) \to \varphi(\xi)$.

In particular, if ξ is a strict local minimum or maximum point of φ, then ξ is C^0-stable critical point.

Let ε be a parameter, which depends on λ, defined as

$$p\lambda \left(-\frac{4}{p} \log \varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{1.7}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$.