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Critical behavior for the polyharmonic operator with Hardy potential

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ABSTRACT

Let us consider the Dirichlet problem

$$\begin{cases} \mathcal{L}_\mu[u] := (-\Delta)^m u - \mu \dfrac{u}{|x|^{2m}} = u^{2^*-1} + \lambda u, \quad u > 0 \text{ in } B \\ D^\beta u|_{\partial B} = 0 \quad \text{for } |\beta| \le m-1 \end{cases}$$

where B is the unit ball in $\mathbb{R}^n$, $n > 2m$, $2^* = 2n/(n-2m)$. We find that, whatever n may be, this problem is critical (in the sense of Pucci–Serrin and Grunau) depending on the value of $\mu \in [0, \overline{\mu})$, $\overline{\mu}$ being the best constant in Rellich inequality. The present work extends to the perturbed operator $(-\Delta)^m - \mu|x|^{-2m} I$ a well-known result by Grunau regarding the polyharmonic operator (see Grunau (1996)).

1. Introduction

The present paper deals with non-existence results for weak solutions to the problem

$$\begin{cases} \mathcal{L}_\mu[u] := (-\Delta)^m u - \mu \dfrac{u}{|x|^{2m}} = u^{2^*-1} + \lambda u, \quad u > 0 \text{ in } B \\ u \in H^m_{0,r}(B) \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^n$, $n \ge 2m+1$ and $H^m_{0,r}(B)$ is the space of the functions $v \in H^m_0(B)$ with spherical symmetry.

Throughout this paper we shall assume that $0 \le \mu < \overline{\mu}$, where $\overline{\mu}$ is the best constant for the Rellich inequality (see the Notations below)

$$\int_{\mathbb{R}^n} |D^m u|^2 \, \mathrm{d}x \ge \overline{\mu} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2m}} \, \mathrm{d}x \quad \forall\, u \in \mathcal{D}^{m,2}(\mathbb{R}^n) \tag{1.2}$$

which is not achieved by any $u \in \mathcal{D}^{m,2}(\mathbb{R}^n)$ (see [3,12]). Being $\mu < \overline{\mu}$, $\mathcal{L}_\mu$ is positive defined.

Let us set

$$P_\mu(z) = (-1)^m \prod_{i=1}^m (z+n-2i)(z+2-2i) - \mu; \tag{1.3}$$

this polynomial will play a crucial role in all our discussion, as it is a sort of "symbol" for $\mathcal{L}_\mu$. We know from [3,12] that

$$\overline{\mu} = P_0(m - n/2) = (-4)^m (1 - m/2 - n/4)^{\overline{m}} (n/4 - m/2)^{\overline{m}} \tag{1.4}$$

where $a^{\overline{h}} := \prod_{j=0}^{h-1} (a+j)$ (see the Notations below).

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The behavior of problem (1.1) is deeply influenced by the amount of μ, and we shall obtain non-existence results depending on μ and λ.

More precisely, let us define

$$\mu_1 := \begin{cases} P_0(-n/2) = (-4)^m(1-n/4)^{\overline{m}}(n/4)^{\overline{m}} & n \geq 4m+1; \\ 0 & 2m+1 \leq n \leq 4m. \end{cases} \tag{1.5}$$

An elementary investigation about $P_0(x)$ for $x \in [2m-n, 0]$ shows that $0 \leq \mu_1 < \overline{\mu}$.

Definition 1. We say that μ is critical for $\mathcal{L}_\mu$ if $\mu_1 < \mu < \overline{\mu}$ when $n \geq 4m$, or $\mu_1 \leq \mu < \overline{\mu}$ when $2m+1 \leq n \leq 4m-1$.

In other words, any $\mu \in [0, \overline{\mu})$ is critical when $2m+1 \leq n \leq 4m-1$; any $\mu \in (\mu_1, \overline{\mu})$ is critical for $n \geq 4m$.

Now we may state our theorem.

Theorem 1. *If μ is critical for $\mathcal{L}_\mu$, then there exists $\lambda_* = \lambda_*(\mu, n) > 0$ such that for $\lambda < \lambda_*$ problem (1.1) admits no nontrivial positive radial weak solutions in $H_0^m(B)$.*

A few words of comment. Theorem 1 generalizes to the case of problem (1.1) the well-known result by Grunau (see [6]) regarding the case $\mu = 0$, i.e. when the linear operator is the polyharmonic operator $(-\Delta)^m$, and indeed, when possible, we have tried to transpose to our case Grunau's argument, which in turn originates from Theorem 1.2$''$ of [1].

In [6] Grunau shows that, when $n = 2m+1 \dots 4m-1$, $(-\Delta)^m$ has a critical behavior, which means that there exists $\lambda_* > 0$ such that the critical problem for $(-\Delta)^m$ has no *positive* radial solutions for $\lambda < \lambda_*$; this was a considerable step forward in proving the well-known conjecture by Pucci–Serrin (see [13]), which states the same claim, but without the restriction of the positivity of u.

Now, if we consider the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^n$, i.e. $|x|^{2m-n}$, we may remark that $|x|^{2m-n}$ belongs to L^2_{loc} iff $n = 2m+1 \dots 4m-1$. In the light of the results of Pucci–Serrin and Grunau, this is not a coincidence: in [7] it is shown for some classes of problems, each class depending on a continuous parameter, that

critical behavior occurs when the (generalized) fundamental solution (depending on the parameter) belongs to L^2_{loc}.

For more detailed motivation of this principle we refer to [7]; what is relevant here is that this principle applies in the present work. To see this, let us remark (see Section 2) that $|x|^\sigma$ solves $\mathcal{L}_\mu[|x|^\sigma] = 0$ in $\mathbb{R}^n \setminus \{0\}$ iff $P_\mu(\sigma) = 0$; now, if we denote by $\beta_1 = \beta_1(\mu)$ the continuous branch among the roots of P_μ which starts from $2m-n$ when $\mu = 0$, we may reasonably call $|x|^{\beta_1}$ the (generalized) fundamental solution of $\mathcal{L}_\mu$. Then it is easy to see that μ is critical in the sense of our Definition 1 iff $\beta_1 > -n/2$, which means that the (generalized) fundamental solution is in L^2_{loc}.

When $m = 1, 2$, the nonlinear critical problem for $\mathcal{L}_\mu$ has been extensively studied in [7,2] respectively, where the analogue of Theorem 1 has been proved in a stronger version: namely it is proved that, when μ is critical, there exist no nontrivial radial solutions u for $\lambda > 0$ sufficiently small, without any assumption about the positivity of u (indeed the theorem in [7] is enounced for positive solutions, but from the proof it is evident that the theorem holds for any radial solution). This is achieved by means of sharp radial Pohozaev identities and, when $m = 2$, suitable Hardy inequalities; this technique does not seem to apply to $\mathcal{L}_\mu$ for general m.

Another remark: many results about critical behavior of nonlinear critical problems state nonexistence theorems of *classical* solutions. But in our case, when $\mu > 0$, we must face singular (hence weak) solutions, which in general have a pole at the origin. This, among other technicalities, leads to state a Pohozaev identity for weak solutions (see Section 4) in a ball.

This paper is organized as follows: in Section 2 we give an explicit representation formula for the solution to the linear problem

$$\begin{cases} \mathcal{L}_\mu[u] = f & \text{in } B, \\ u \in H^m_{0,r}(B) \end{cases} \tag{1.6}$$

in terms of the roots of the polynomial $P_\mu(z)$. Section 3 is devoted to the study of the auxiliary function w_μ, which solves the problem

$$\begin{cases} \mathcal{L}_\mu[w_\mu] = 0 & \text{in } B \\ w_\mu \in H^m_r(B) \\ w_\mu(1) = \cdots = w_\mu^{(m-2)}(1) = 0, & w_\mu^{(m-1)}(1) = (-1)^{m-1}, \end{cases} \tag{1.7}$$

where $H^m_r(B)$ is the closed subspace of the functions of $H^m(B)$ with spherical symmetry; by coupling problems (1.6) and (1.7) we shall get useful estimates about u when f is the right hand side of problem (1.1).

In Section 4 we derive a Pohozaev identity for weak solutions to (1.1), and finally in Section 5 we collect together all the informations, so proving Theorem 1.