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Jump discontinuous viscosity solutions to second order degenerate elliptic equations

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ABSTRACT

We prove existence and uniqueness of viscosity solutions to second order degenerate elliptic equations which are continuous or jump discontinuous on some interior surface. Moreover, we derive sharp conditions for Lipschitz regularity or gradient blow up of the viscosity solutions on such a surface.

1. Introduction and model equation

Let us consider the following simple example [15]

$$\begin{aligned} &-u''(x) - 18x(u'(x))^4 = 0 \quad \text{in } (-1, 1) \\ &u(-1) = -B, \qquad u(1) = B, \quad B \geq 0. \end{aligned} \tag{1.1}$$

One can easily check that for $0 < B < 1$ problem (1.1) has a unique classical solution $u(x) \in C^2([-1, 1])$. However, for $B = 1$ the unique viscosity solution is $u(x) = \sqrt[3]{x}$, while for $B > 1$ there is no longer a continuous viscosity solution of (1.1).

The functions $\tilde{u}$ and $\tilde{v}$ given by

$$\tilde{u} = \begin{cases} \sqrt[3]{x} - 1 + B & \text{for } 0 \leq x \leq 1 \\ \sqrt[3]{x} + 1 - B & \text{for } -1 \leq x < 0 \end{cases} \qquad \tilde{v} = \begin{cases} \sqrt[3]{x} - 1 + B & \text{for } 0 < x \leq 1 \\ \sqrt[3]{x} + 1 - B & \text{for } -1 \leq x \leq 0 \end{cases}$$

differ only in the origin, $\tilde{u} \in USC([-1, 1])$, $\tilde{v} \in LSC([-1, 1])$, are viscosity sub- and supersolutions of (1.1) for $B > 1$. They are equal at the boundary, but $\sup_{[-1,1]}(\tilde{u} - \tilde{v}) = \tilde{u}(0) - \tilde{v}(0) = 2(B - 1) > 0$ and the comparison principle fails for problem (1.1) (see Fig. 1).

From the theory of viscosity solutions [9] it is well known that the Perron method, without comparison principle for the semicontinuous viscosity sub- and supersolutions, can in general produce discontinuous solutions. For the time being, there

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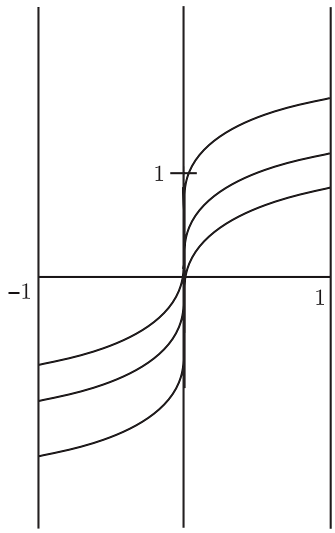

Fig. 1. Solutions with a transition layer and discontinuity at zero.

is a notion of discontinuous viscosity solutions for second order elliptic equations which says that u is a (discontinuous) viscosity solution of

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1.2}$$

$$u = \psi \quad \text{on } \partial\Omega \tag{1.3}$$

iff u^* is a viscosity subsolution and u_* is a viscosity supersolution of (1.2) (see [1,4,3], [10, Sec. 6]). Here u^* and u_* are the upper and lower semicontinuous envelopes of u defined by

$$u_*(x) = \lim_{\varepsilon\downarrow 0} \inf\{u(y) : |x-y| \le \varepsilon\}, \qquad u^*(x) = \lim_{\varepsilon\downarrow 0} \sup\{u(y) : |x-y| \le \varepsilon\}.$$

According to this definition the function $u_a(x) = \sqrt[3]{x} - 1 + B$ for $0 < x \le 1$, $u_a(x) = \sqrt[3]{x} + 1 - B$ for $-1 \le x < 0$ and $u_a(0) = a$ $(B > 1, 1 - B < a < B - 1)$ is a (discontinuous) viscosity solution for every such choice of a. In fact $(u_a)^* = \tilde{u}$, $(u_a)_* = \tilde{v}$ and as noted above they are viscosity sub- and supersolution.

Unfortunately problem (1.1) has infinitely many viscosity solutions u_a, namely one for every $a \in [1 - B, B - 1]$ satisfying $u_a(0) = a$. In order to overcome the difficulties with the non uniqueness of the notion of discontinuous viscosity solutions our first approach in [14] was to define the viscosity solution as a multivalued function adding the whole interval $[-B + 1, B - 1]$ at the origin for problem (1.1). In this way all problems with the validity of the comparison principle, or non uniqueness of the solution will be overcome. The heuristic argument is to rotate the coordinate system in $\pi/2$ so that the x-variable to be the new dependent variable and u becomes the new independent variable. Now the function $x(u)$ is well defined and flat near the origin. Unfortunately, the calculations are very complicated and only the mean curvature operator is invariant. Moreover, this solution is not stable under small perturbations of the coefficients of the equation.

In order to motivate our second approach in [16] let us consider the same example (1.1) when $B < 0$. For

$$\int_{-1}^{1} (27(1-t^2))^{-\frac{1}{3}} dt =: -2B_* \ge -2B$$

(1.1) has a unique classical solution $u \in C^2(-1, 1) \cap C([-1, 1])$ which satisfies the boundary data in classical sense

$$u(x) = -B + \int_{-1}^{x} [k^{-3} - 27(1-t^2)]^{-\frac{1}{3}} dt, \quad k = u'(-1).$$

If $-B > -B_*$ then (1.1) has only viscosity solutions with infinite slope on the boundary

$$u_C(x) = C - \int_{-1}^{x} [27(1-t^2)]^{-\frac{1}{3}} dt, \quad C = \text{const}, \quad -B \ge C \ge B$$

(see Fig. 2).

More precisely $u_C(x) \in C^2(-1, 1) \cap C([-1, 1])$, but the solution satisfies the Dirichlet conditions only in weak (viscosity) sense (see Section 7 in [9]).

Note that the comparison principle still holds for monotone decreasing viscosity sub- and supersolutions from Theorem 2.6 in [15] (see also [2]).