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On the existence of positive solutions and solutions with compact support for a spectral nonlinear elliptic problem with strong absorption

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ABSTRACT

We study a semilinear elliptic equation with a strong absorption term given by a non-Lipschitz function. The motivation is related with study of the linear Schrödinger equation with an infinite well potential. We start by proving a general existence result for non-negative solutions. We use also variational methods, more precisely Nehari manifolds, to prove that for any $\lambda > \lambda_1$ (the first eigenvalue of the Laplacian operator) there exists (at least) a non-negative solution. These solutions bifurcate from infinity at λ_1 and we obtain some interesting additional information. We sketch also an asymptotic bifurcation approach, in particular this shows that there exists an unbounded continuum of non-negative solutions bifurcating from infinity at $\lambda = \lambda_1$. We prove that for *some* neighborhood of $(\lambda_1, +\infty)$ the positive solutions are unique. Then *Pohozaev's identity* is introduced and we study the existence (or not) of free boundary solutions and compact support solutions. We obtain several properties of the energy functional and associated quantities for the ground states, together with asymptotic estimates in λ, mostly for $\lambda \nearrow \lambda_1$. We also consider the existence of solutions with compact support in Ω for λ large enough.

1. Introduction

We study in this paper the existence of different kinds of non-negative solutions to the semilinear elliptic equation

$$P(q,\alpha,\lambda) = \begin{cases} -\Delta u + q(x)|u|^{\alpha-1}u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 1$), λ is a real parameter, $0 < \alpha < 1$ and $q(x) \geq 0$ is a real function satisfying suitable assumptions.

The equation $P(q, \alpha, \lambda)$ is a typical example of the so-called diffusion–reaction equations. If $q \equiv 1$ and $\alpha > 1$ it is the well-known logistic equation in population dynamics. In this case there is a unique positive solution u_λ for any $\lambda > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions and it follows immediately from the Strong Maximum Principle that if $u_\lambda \geq 0$, $u_\lambda \not\equiv 0$, is a solution then $u_\lambda > 0$ in Ω and $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, i.e., all non-negative solutions to the logistic equation are actually positive.

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But this is not the case for $P(q, \alpha, \lambda)$ for $q \equiv 1$ and $0 < \alpha < 1$ and for some problems of this kind previously studied. The semilinear problem

$$\begin{cases} -u'' + u^m = \lambda u^q & \text{in } (-1, 1), \\ u(\pm 1) = 0, \end{cases} \tag{2}$$

with $0 < m < q < 1$ was studied in [12], where it was proved by using energy methods for ODEs that there exist two critical values $0 < \lambda^* < \lambda^{**}$ such that: (i) for $0 < \lambda < \lambda^*$ there is no solution to (2); (ii) for any $\lambda > \lambda^*$ there is an upper branch of positive solutions $u_\lambda > 0$ with $\frac{\partial u_\lambda}{\partial \nu}(\pm 1) < 0$; (iii) for $\lambda^* < \lambda < \lambda^{**}$ there is a lower branch $v_\lambda > 0\,(0 < v_\lambda < u_\lambda$ on $(-1, 1))$ with $\partial v_\lambda / \partial n(\pm 1) < 0$; (iv) for $\lambda = \lambda^{**}$ there is a solution $v_{\lambda^{**}} > 0$ such that $\partial v_{\lambda^{**}} / \partial n(\pm 1) = 0$; (v) from this solution, which is also a solution of Eq. (2) on the whole real line, it is possible to build by stretching, gluing and rescaling continua of infinitely many compact support solutions, whose precise description is given in [12]. These results were extended in [14] to the singular case $-1 < m < q < p - 1$ and to the p-Laplacian as well. The corresponding N-dimensional problem to (2) was studied for a star-shaped bounded domain Ω in [18] by using a combination of Pohozaev's identity and variational methods obtaining similar but less precise results.

The motivation for studying later problem $P(q, \alpha, \lambda)$ with $q \equiv 1$ and $0 < \alpha < 1$ arises from general observations of the first author [10,11] concerning solutions for the linear Schrödinger equation

$$\begin{cases} -u'' + V(x)u = \lambda u & \text{in } (-R, R), \\ u(\pm R) = 0, \end{cases} \tag{3}$$

for a given $R > 0$ or on the real line, where $V(x)$ is the so-called infinite well potential, introduced by Gamow. It turns out that there is some ambiguity in the treatment of the case of the real line: what is mentioned to be solutions in most of the text-books are not classical solutions since Dirac's deltas appear, and they are solutions in the sense of distributions but of a different equation where deltas are included. In this situation solutions of the semilinear equation $P(q, \alpha, \lambda)$ provide some kind of, say, "alternative approach" [10,11].

The one-dimensional case $\Omega = (-R, R)$ was studied in detail in [13] by using the same phase plane methods in ODEs. There it was proved that for $0 < \lambda < \lambda_1$ there is no solution to (1) (with $q = 1$) and for $\lambda_1 < \lambda < \lambda^*$, where λ^* is a critical value given explicitly, there is a unique solution $u_\lambda > 0$ with $\partial u_\lambda / \partial n(\pm R) < 0$ bifurcating at infinity for $\lambda = \lambda_1$. Moreover u_λ is decreasing as a function of λ and, as in the preceding example, the solution $u_{\lambda^*} > 0$ in $(-R, R)$ is such that $u'_{\lambda^*}(\pm R) = 0$ and from this free boundary solution it is possible, once again, to build continua of compact support solutions. We also obtain the asymptotic behavior (in λ) of the solutions, namely

$$\|u_\lambda\|_{L^\infty(-R,R)} \leq \frac{C}{\lambda^{1-\alpha}}.$$

Thus we have obtained that $\lambda^* > 0$ and $u_{\lambda^*} > 0$ are the first eigenvalue and its corresponding eigenfunction for the linear eigenvalue problem

$$\begin{cases} -w'' + |u_{\lambda^*}|^{\alpha - 1}\, w = \lambda w & \text{in } (-R, R), \\ w(\pm R) = 0. \end{cases} \tag{4}$$

and that $u'_{\lambda^*}(\pm R) = 0$. It is in this sense that we have an "alternative approach" for solutions to the linear Schrödinger equation.

Problem $P(q, \alpha, \lambda)$ is studied in [16] for $q = 1$ and $0 < \alpha < 1$ as a particular case of a much more general class of problems allowing more general nonlinear terms and boundary conditions. The main result in [16] is the existence of an unbounded continuum of non-negative solutions bifurcating from infinity at the asymptotic bifurcation point λ_1. The method of proof was to apply global asymptotic bifurcation theorems by Rabinowitz [31] by using as a tool some theorem in [8]. More details are given below.

Existence of a weak non-negative solution for any $\lambda > \lambda_1$ was obtained later by Porretta [28] this time by using variational methods, namely a variant of the Mountain Pass Theorem. There were some complementary results concerning, for example, estimates for the norm of the solutions, but the problem of the existence (or not) of positive solutions was not considered.

In this paper we deal first with general existence results for non-negative solutions to $P(q, \alpha, \lambda)$. First, in Section 2, we use variational methods, more precisely Nehari manifolds [4,19,18,34] and prove that for any $\lambda > \lambda_1$ there exists (at least) a non-negative solution. These solutions bifurcate from infinity at λ_1 and we obtain some interesting additional information. In a second part of Section 2 we sketch the asymptotic bifurcation approach above mentioned, in particular this shows that there exists an unbounded continuum of non-negative solutions bifurcating from infinity at $\lambda = \lambda_1$. In Section 3 we study different kinds of solutions. It is possible to show that solutions u_λ bifurcating from infinity at $\lambda = \lambda_1$ are $u_\lambda > 0$ with $\partial u_\lambda / \partial n < 0$ on $\partial\Omega$ for *some* neighborhood of $(\lambda_1, +\infty)$. Under some additional assumptions it is also possible to show, by using the results in [17], that positive solutions are unique there. Then *Pohozaev's identity* is introduced and here the coefficient $q(x)$ plays an interesting role concerning existence (or not) of a free boundary and compact support solutions.

In Section 4 we collect several results concerning regularity and differentiability properties of the energy functional and associated quantities for the ground states, together with asymptotic estimates in λ, mostly for $\lambda \searrow \lambda_1$. The existence of solutions with compact support in Ω is considered in Section 5. With the usual philosophy of reaction–diffusion equations