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Criteria for the regularity of the solutions to the Navier–Stokes equations based on the velocity gradient

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ABSTRACT

We study the regularity of solutions to the Navier–Stokes equations in the whole three-dimensional space under the assumption that some additional conditions are imposed on one or more entries of the velocity gradient. Many such results with conditions using the Lebesgue spaces can be found in the literature, starting with the classical Beirao da Veiga's result. The main goal of the present paper is to generalize the known results by replacing the standard Lebesgue spaces by wider Besov spaces in space variables. In most such cases the technique of the proof leads to the deterioration of the Prodi–Serrin scale, so the results with the Besov spaces are not still quite satisfactory. Nevertheless, our technique leads to extensions and improvements of some results from the literature.

1. Introduction

This paper contains an improvement of several regularity criteria for the Navier–Stokes equations containing one or more entries of the velocity gradient and using the Besov spaces. It combines methods developed in [9,5,11,12,25,26].

We consider the Navier–Stokes equations in the whole three-dimensional space, i.e.

$$\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{2}$$

$$u|_{t=0} = u_0, \tag{3}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity and pressure and $u_0 = u_0(x) = (u_{01}(x), u_{02}(x), u_{03}(x))$ is a given initial velocity.

It is known that for $u_0 \in L^2_\sigma(\mathbf{R}^3)$ (solenoidal functions from $L^2(\mathbf{R}^3)$) the problem (1)–(3) possesses at least one global weak solution u satisfying the energy inequality $\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \, d\tau \le \|u_0\|_2^2$ for every $t \ge 0$ (see [16] or [32]).

If $u_0 \in W^{1,2}_\sigma(\mathbf{R}^3)$ (solenoidal functions from the standard Sobolev space $W^{1,2}(\mathbf{R}^3)$) then there exists a global weak solution u satisfying the energy inequality and regular on $(0,T^*)$, $T^* > 0$, i.e. $\nabla u \in L^\infty(0,T^*; L^2(\mathbf{R}^3))$, $u \in L^2(0,T^*; W^{2,2}(\mathbf{R}^3))$ and (subsequently) $u \in C^{\infty_{loc}}((0,T^*) \times R^3)$ (see [32]). T^* depends on $W^{1,2}(\mathbf{R}^3)$-norm of u_0 and may possibly be small. In the following text we deal with the weak solutions described in this paragraph.

It is a classical question to ask whether u is regular on $(0,T)$ for every $T > 0$ (a globally regular solution) or whether there exists a finite time blow-up, i.e. whether there exists $T < \infty$ such that $\lim_{t\to T_-} \|\nabla u(t)\|_2 = \infty$. This outstanding problem

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was firstly formulated by J. Leray in his pioneering paper from 1934 (see [16]). Despite the efforts of many mathematicians the answer still remains unanswered and its solution seems to be beyond the scope of the present techniques. Nevertheless, many partial results have been presented in the last fifty years: it is known that the solutions are globally regular in the case of small initial data or regular on a small time interval under the assumption of some sufficient smoothness of the initial conditions (see [32]).

Further, some additional regularity conditions imposed on the weak solution can insure its regularity. The first result in this direction is usually referred to as the Prodi–Serrin conditions: if a weak solution satisfies additionally the integrability properties $u \in L^t(0, T; L^s(\mathbf{R}^3))$, $T > 0$, where $2/t + 3/s = 1$ and $s \in [3, \infty]$, then u is regular on $(0, T)$ (see [27,28] for $s > 3$ and [10] for $s = 3$). We will quote such a 1-scaled criterion of regularity as

$$u \in (PS)_1, \qquad s \in [3, \infty]. \tag{4}$$

In [2] a classical result concerning the velocity gradient was proved, namely, the additional condition

$$\nabla u \in (PS)_2, \qquad s \in [3/2, \infty] \tag{5}$$

implies the regularity of u. Notice, that these results are optimal from the scaling point of view. They can be further extended by the use of the nonhomogeneous Besov spaces: let $\omega = rot\, u \in L^t(0, T; B^{-3/s}_{\infty,\infty}(\mathbf{R}^3))$, where $2/t+3/s = 2$ and $s \in (3/2, \infty)$, then u is regular (see [8,7,30]). We will quote such a regularity criterion as

$$\omega \in (PSB)_2, \qquad s \in (3/2, \infty), \tag{6}$$

by which we stress that we work with the Besov spaces. Since $L^s(\mathbf{R}^3) \hookrightarrow B^0_{s,\infty}(\mathbf{R}^3) \hookrightarrow B^{-3/s}_{\infty,\infty}(\mathbf{R}^3),\ s \in [1, \infty]$, the criterion (6) constitutes a generalization of criteria (4) and (5). A slightly different criterion can also be proved: u is regular if

$$\partial_i u_j \in (PSB)_2, \qquad s_{i,j} \in (3/2, \infty), \quad i, j = 1, 2, 3. \tag{7}$$

Let us note that the following boundary case corresponding to the above mentioned result from [10] has not yet been fully solved: if $u \in L^\infty(0, T; B^{-1}_{\infty,\infty})$ then the regularity of u is known only under the additional assumption that the jumps of u on $(0, T)$ in the $B^{-1}_{\infty,\infty}$-norm are sufficiently small (that is, for example, if $u \in C((0, T), B^{-1}_{\infty,\infty})$)—see [8].

In 1999 J. Neustupa and P. Penel came with a new interesting approach: they asked an original question if the velocity components are coupled in such a way that the regularity of one of them implies the regularity of all velocity components. They were able to prove a positive answer in [19] showing so that the only possible epochs of singularity of solutions concern simultaneously all three velocity components. The same authors further extended their results in [20,17,21] proving the regularity under the assumptions of the validity of some very inspiring geometric conditions. Since then a plethora of follow-up results have been published showing that various additional conditions imposed on one or two components of velocity, one or more entries of the velocity gradient or possibly on the pressure (and/or other variables) ensure the regularity of the solution (see [2,3,5,4,13,14,23,34] and the results cited there). But while the criteria mentioned in the previous paragraph (i.e. with additional regularity conditions imposed on all components of the velocity or on all items of the velocity gradient) are optimal from the scaling point of view, it is generally not the case for the criteria where the regularity conditions are imposed only on some velocity components or on some items of the velocity gradient.

With regard to the goal of this paper we will focus from now on only on some criteria with conditions imposed on one or several items of the velocity gradient: it was proved in [23] that u is regular if $\partial_j u_j \in (PS)_2,\ s \in (3/2, \infty],\ j = 2, 3$ and this criterion is optimal from the scaling point of view. It is not the case for the gradient of one velocity component, where the optimal criterion $\nabla u_3 \in (PS)_2$ is known only for $s \in (9/5, 2)$. For the proof of this criterion one can use the fact that $L^p(\mathbf{R}^3)$ is continuously embedded into $\dot{H}^{3/2-3/p}(\mathbf{R}^3),\ p \in (1, 2]$ and then apply the following magnificent result which has been recently proved by Chemin and Zhang in [7] and which constitutes a breakthrough in the investigation of ∇u_3: u is regular on $(0, T)$ if $u_3 \in L^p(0, T; \dot{H}^{1/2+2/p}(\mathbf{R}^3))$, where $p \in (4, 6)$. $\dot{H}^q(\mathbf{R}^3)$ denotes a standard homogeneous Sobolev space (see [1]). To prove this criterion the authors wrote the Navier–Stokes system in terms of two new unknowns: the third component of the vorticity $\partial_1 u_2 - \partial_2 u_1$ and the quantity $\partial_3 v_3$ and they were able to show subtle estimates of these two variables by the use of the anisotropic Besov spaces. They finished the proof using the criterion (7). The best criteria for ∇u_3 with $s \in (3/2, 9/5] \cup [2, \infty)$ known so far are not optimal—see [25,31,29], see also Theorem 1 in Section 3 for the precise information. In the case of $\partial_3 u$ the optimal result $\partial_3 u \in (PS)_2$ has been proved in [15] for $s \in [9/4, 3]$ and in [3] for $s \in [27/16, 9/4)$. For $s \in (3/2, 27/16) \cup (3, \infty)$ the present results are not optimal. We will finish this short survey by the discussion of the criteria in which the additional regularity condition is imposed only on one item of the velocity gradient. Firstly, the best result concerning $\partial_3 u_3$ has so far been reached in [13,12], namely

$$\partial_3 u_3 \in (PS)_{\frac{-7s+12+\sqrt{289s^2-264s+144}}{8s}}, \qquad s \in (9/5, \infty).$$

The result is not optimal for any s, for $s \to 9/5_+$ one approaches the best Prodi–Serrin scale, $5/3$. Secondly, for the case of $\partial_1 u_3$ the best result proved so far comes from [13,12]:

$$\partial_3 u_3 \in (PS)_{\frac{-9s+3+\sqrt{103s^2-12s+9}}{2s}}, \qquad s \in (3, \infty).$$