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Non-uniform elliptic equations in convex Lipschitz domains

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ABSTRACT

Non-uniform elliptic equations in convex Lipschitz domains are concerned. The non-smooth domains consist of a periodic connected high permeability sub-region and a periodic disconnected matrix block subset with low permeability. Let $\epsilon \in (0, 1]$ denote the size ratio of the matrix blocks to the whole domain and let $\omega^2 \in (0, 1]$ denote the permeability ratio of the disconnected matrix block subset to the connected sub-region. The $W^{1,p}$ norm for $p \in (1, \infty)$ of the elliptic solutions in the high permeability sub-region is shown to be bounded uniformly in ω, ϵ. However, the $W^{1,p}$ norm of the solutions in the low permeability subset may not be bounded uniformly in ω, ϵ. Roughly speaking, if the sources in the low permeability subset are small enough, the solutions in that subset are bounded uniformly in ω, ϵ. Otherwise the solutions cannot be bounded uniformly in ω, ϵ. Relations between the sources and the variation of the solutions in the low permeability subset are also presented in this work.

1. Introduction

Uniform L^p gradient estimate for the solutions of non-uniform elliptic equations in bounded convex Lipschitz domains is presented. Let Ω be a bounded domain in $\mathbb{R}^n$ for $n \geq 2$, $\partial\Omega$ denote the boundary of Ω, $\epsilon \in (0, 1]$, $\Omega(2\epsilon) \equiv \{x \in \Omega : dist(x, \partial\Omega) > 2\epsilon\}$, $Y \equiv (0, 1)^n$ consist of a smooth sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y_m})$, $\Omega_m^\epsilon \equiv \{x : x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon)$ for some $j \in \mathbb{Z}^n\}$ be a disconnected subset of Ω, Ω_f^ϵ $(\equiv \Omega \setminus \overline{\Omega_m^\epsilon})$ represent a connected sub-region of Ω, and $\mathbf{K}_{\nu,\epsilon}(x) \equiv \begin{cases} 1 & \text{if } x \in \Omega_f^\epsilon \\ \nu & \text{if } x \in \Omega_m^\epsilon \end{cases}$ for any $\nu > 0$. The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla U + G) = F & \text{in } \Omega, \\ (\mathbf{K}_{\omega^2,\epsilon} \nabla U + G) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \Pi_\epsilon U|_{\Omega_f^\epsilon} dx = 0, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0, 1]$, $\vec{\mathbf{n}}$ is a unit normal vector on $\partial\Omega$, and G, F are given functions. Π_ϵ in (1.1) is an extension operator (see [1] or Lemma 2.1) and $\Pi_\epsilon U|_{\Omega_f^\epsilon}$ is the extension function of $U|_{\Omega_f^\epsilon}$ in Ω. The problem has applications in heat transfer in two-phase media, flows in highly heterogeneous media, the stress in composite materials, and so on (see [3,11,14] and references therein). If G, F are bounded in Ω and $\int_\Omega F dx = 0$, a solution of (1.1) in Hilbert space $H^1(\Omega)$ exists uniquely for each ω, ϵ by Lax–Milgram Theorem [10]. The L^2 norm of the gradient of the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in $\omega, \epsilon \in (0, 1]$ if the sources G, F are small in Ω_m^ϵ. However, the L^2 norm of the gradient of the

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solution of (1.1) in matrix blocks Ω_m^ϵ can be very large when ω closes to 0. It is interested to ask whether the uniform bound in ω, ϵ for the gradient of the solution of (1.1) can be extended to L^p space for any $p \in (1, \infty)$ or not.

$W^{1,p}$ estimate and Lipschitz estimate uniform in ϵ for the Laplace equation in periodic perforated domains were derived in [15,18]. For uniform elliptic equations with Dirichlet boundary condition and with discontinuous or periodic oscillatory coefficients, the uniform bound in ϵ for $W^{k,p}$ norm or for Lipschitz norm in the whole domain could be found in [4,5,7,14,16,20]. For example, Lipschitz estimate and $W^{2,p}$ estimate for uniform elliptic equations with discontinuous coefficients had been proved in [14,16]. Uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ϵ for uniform elliptic equations with Hölder periodic coefficients were shown in [4,5]. Uniform $W^{1,p}$ estimate in ϵ for uniform elliptic equations with continuous or with VMO periodic coefficients were considered in [7,20].

For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [12]. Uniform Hölder and Lipschitz estimates in ω, ϵ for (1.1) $_1$ with Dirichlet boundary condition were shown in [22]. Here we consider the non-uniform elliptic equations in Lipschitz domains. It is proved that $W^{1,p}$ norm for the solution of (1.1) in the connected sub-region Ω_f^ϵ is bounded uniformly in ω, ϵ under some proper assumptions. But, the $W^{1,p}$ norm for the solution of (1.1) in the disconnected subset Ω_m^ϵ may not be bounded uniformly in ω, ϵ. If the sources G, F in the low permeability subset Ω_m^ϵ are very small, the solutions in Ω_m^ϵ are still bounded uniformly in ω, ϵ like the solutions of uniform elliptic equations. If the sources are not small enough, the solutions in Ω_m^ϵ cannot be bounded uniformly in ω, ϵ again.

The rest of this work is organized as follows: Notation and main result are stated in Section 2. In Section 3, we present a priori estimates for interface problems and present uniform Hölder, uniform Lipschitz, and uniform $W^{1,p}$ local estimates in ω, ϵ for the solutions of non-uniform elliptic equations in periodic domains. The proof of the main result is given in Section 4. In Section 5, we show the uniform Hölder and the uniform Lipschitz estimates in ω, ϵ for non-uniform elliptic equations, claimed in Section 3. In Appendix, we give a proof of Theorem 4.1, which is a modification of Theorem 3.3 in [19].

2. Notation and main result

Let $C^{k,\alpha}$ denote the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, $W^{s,p}$ denote the Sobolev space with norm $\|\cdot\|_{W^{s,p}}$, and $[\varphi]_{C^{0,\alpha}}$ be the Hölder semi-norm of φ for $k \geq 0, \alpha \in [0, 1], s \geq -1, p \in [1, \infty]$ (see [2,10]). $L^p = W^{0,p}$ and $H^1 = W^{1,2}$. $C^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions in $\mathbb{R}^n$, $C_0^\infty(D)$ is a subset of $C^\infty(\mathbb{R}^n)$ with support in D, and $C_{per}^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable Y-periodic functions in $\mathbb{R}^n$. $W_0^{s,p}(D)$ is the closure of $C_0^\infty(D)$ under the $W^{s,p}$ norm and $W_{per}^{s,p}(\mathbb{R}^n)$ is the closure of $C_{per}^\infty(\mathbb{R}^n)$ under the $W^{s,p}$ norm and $\|\varphi\|_{W_{per}^{s,p}(\mathbb{R}^n)} \equiv \|\varphi\|_{W^{s,p}(Y)}$ for $s \geq 1, p \in [1, \infty]$. $\mathcal{A}_m \equiv \{x \in \mathbb{R}^n : x \in Y_m + j \text{ for some } j \in \mathbb{Z}^n\}$ and $\mathcal{A}_f \equiv \mathbb{R}^n \setminus \overline{\mathcal{A}_m}$. $\mathcal{H}_{per}^1(\mathbb{R}^n) \equiv \{\varphi \in W_{per}^{1,2}(\mathbb{R}^n) : \int_{Y_f} \varphi(y)dy = 0\}$ and $\mathcal{H}_{per}^1(\mathcal{A}_f) \equiv \{\varphi|_{\mathcal{A}_f} : \varphi \in \mathcal{H}_{per}^1(\mathbb{R}^n)\}$. Let $\|\varphi_1, \ldots, \varphi_m\|_{\mathbf{B}_1} \equiv \|\varphi_1\|_{\mathbf{B}_1} + \cdots + \|\varphi_m\|_{\mathbf{B}_1}$, $\|\varphi\|_{\mathbf{B}_1 \cap \mathbf{B}_2} \equiv \|\varphi\|_{\mathbf{B}_1} + \|\varphi\|_{\mathbf{B}_2}$, $B_r(x)$ denote a ball centered at x with radius r, $\overline{D}$ be the closure of D, ∂D be the boundary of D, $|D|$ be the volume of D, $\mathcal{X}_D$ be the characteristic function on D, and $D/r \equiv \{x : rx \in D\}$. For any $\varphi \in L^1(D)$,

$$(\varphi)_D \equiv \fint_D \varphi(y)dy \equiv \frac{1}{|D|}\int_D \varphi(y)dy.$$

$\mathbb{K}_{\omega,1/r} \equiv \begin{cases} 1 & \text{in } \mathcal{A}_f/r \\ \omega & \text{in } \mathcal{A}_m/r \end{cases}$ and $\breve{\mathbf{K}}_{\omega,\nu,r} \equiv \begin{cases} 1 & \text{in } \Omega_f^\nu/r \\ \omega & \text{in } \Omega_m^\nu/r \end{cases}$ for $\omega \in [0, 1]$, $\nu, r \in (0, \infty)$. If $\vec{\mathbf{n}}_y$ is an outward normal vector on ∂Y_m, we define, for any function φ in Y and $x \in \partial Y_m$,

$$\varphi_{,\pm}(x) \equiv \lim_{t \to 0^+} \varphi(x \pm t\vec{\mathbf{n}}_y), \qquad \lfloor \varphi \rfloor(x) = \varphi_{,+}(x) - \varphi_{,-}(x). \tag{2.1}$$

Our main results are:

Theorem 2.1. *Suppose*

A1. *Ω is a bounded convex Lipschitz domain in $\mathbb{R}^n$ for $n \geq 2$,*
A2. *Y_m is a smooth simply-connected sub-domain of Y,*
A3. *$\omega, \epsilon \in (0, 1], \sigma \in [0, 2],\ p \in (1, \infty), G \in L^p(\Omega), F \in W^{-1,p}(\Omega), \langle F, 1\rangle_\Omega = 0$,*

then a $W^{1,p}(\Omega)$ solution of (1.1) exists uniquely and satisfies

$$\begin{cases} \|\mathbf{K}_{\omega^\sigma/\epsilon,\epsilon} U, \mathbf{K}_{\omega^\sigma,\epsilon} \nabla U\|_{L^p(\Omega)} \leq c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon} G\|_{L^p(\Omega)} + \|F\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \dfrac{\omega^\sigma}{\epsilon} \leq 1, \\ \|U, \mathbf{K}_{\omega^\sigma,\epsilon} \nabla U\|_{L^p(\Omega)} \leq c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon} G\|_{L^p(\Omega)} + \|F\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2}\|F\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \dfrac{\omega^\sigma}{\epsilon} \geq 1, \end{cases} \tag{2.2}$$

where c is a constant independent of ω, ϵ, σ. Here $\langle F, 1\rangle_\Omega = 0$ means $\int_\Omega F dx = 0$ in distribution sense.

By energy method and Poincaré inequality [10], we easily get (2.2) for $\sigma = 1, p = 2$ case. But it is not clear whether ∇U is bounded uniformly in $L^2(\Omega_m^\epsilon)$. From Theorem 2.1, we know that if the right hand side of (2.2) is bounded independent of ω, ϵ, σ, then the $W^{1,p}$ norm of the solution U in Ω_f^ϵ is bounded uniformly in ω, ϵ, σ for any $p \in (1, \infty)$. However, the $W^{1,p}$ norm of the solution U in Ω_m^ϵ may not be bounded uniformly in ω, ϵ, σ. From the proof of Theorem 2.1, we see that if the right hand side of (2.2) is uniformly bounded in ω, ϵ, σ, then