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A Global Compactness type result for Palais–Smale sequences in fractional Sobolev spaces

Giampiero Palatucci [a,b,*], Adriano Pisante [c]

[a] Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Campus - Parco Area delle Scienze, 53/a, 43124 Parma, Italy

[b] SISSA, Via Bonomea, 256, 34136 Trieste, Italy

[c] Dipartimento di Matematica, Sapienza Università di Roma, P. le Aldo Moro, 5, 00185 Roma, Italy

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ABSTRACT

We extend the global compactness result by Struwe (1984) to any fractional Sobolev spaces $\dot{H}^s(\Omega)$, for $0 < s < N/2$ and $\Omega \subset \mathbb{R}^N$ a bounded domain with smooth boundary. The proof is a simple direct consequence of the so-called *profile decomposition* of Gérard (1998).

1. Introduction

Since the seminal paper [17], global compactness properties for Palais–Smale sequences in the Sobolev space H^1 have become very important tools in Nonlinear Analysis which have been crucial in many existence results, e.g. for ground states and blow-up solutions for nonlinear Schrödinger equations, for solutions of Yamabe-type equations both in conformal and in CR geometry, for prescribing Q-curvature problems, etc. Together with the aforementioned examples concerning single equations in a scalar unknown function, more difficult systems of PDEs, often related to other geometric problems, share similar compactness properties for their solutions; for instance, this is the case for parametric surfaces of constant mean curvature, harmonic maps from Riemann surfaces into Riemannian manifolds, Yang–Mills connections over four-manifolds, pseudo-holomorphic curves into symplectic manifolds, planar Toda systems, etc. The involved literature is really too wide to attempt any reasonable account here.

In the present note, we aim to extend the global compactness result by M. Struwe for semilinear elliptic equation in H^1_0 to the case of fractional Sobolev spaces $\dot{H}^s$ of any real differentiability order $0 < s < N/2$ by means of the so-called *profile decomposition* first obtained in [9] (see also [13] for an alternative slightly simpler and more abstract approach).

* Corresponding author at: Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Campus - Parco Area delle Scienze, 53/a, 43124 Parma, Italy.

E-mail addresses: giampiero.palatucci@unipr.it (G. Palatucci), pisante@mat.uniroma1.it (A. Pisante).

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Let $N \geq 1$ and for each $0 < s < N/2$ denote by $\dot{H}^s(\mathbb{R}^N)$ the usual L^2-based homogeneous fractional Sobolev spaces[1] defined via Fourier transform as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi. \tag{1.1}$$

In view of the well known critical Sobolev embedding $\dot{H}^s \hookrightarrow L^{2^*}$, where $2^* = 2N/(N-2s)$ is the critical Sobolev exponent, one has equivalently,

$$\dot{H}^s(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) \text{ s.t. } (-\Delta)^{s/2} u \in L^2(\mathbb{R}^N) \right\},$$

where, by definition, $((-\Delta)^{s/2}u)\widehat{\ }(\xi) := |\xi|^s \hat{u}(\xi)$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq 1$, and define the homogeneous Sobolev space $\dot{H}^s(\Omega)$ as the completion of $C_0^\infty(\Omega)$ under the norm (1.1), hence a closed subspace of $\dot{H}^s(\mathbb{R}^N)$. Thus, we have a well defined fractional Laplacian $(-\Delta)^s : \dot{H}^s(\Omega) \to (\dot{H}^s(\Omega))'$ which is a bounded linear operator (isomorphism) given by $((-\Delta)^s u)\widehat{\ }(\xi) := |\xi|^{2s}\hat{u}(\xi)$, so that $\langle v, (-\Delta)^s u \rangle_{H,H'} = (v, u)_H$ for any $u, v \in H = \dot{H}^s(\Omega)$.

Since in the entire space $\dot{H}^s \hookrightarrow L^2_{\mathrm{loc}}$ with compact embedding, we also have $\dot{H}^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (\dot{H}^s(\Omega))'$, both with compact embedding. As a consequence, there is a well defined first eigenvalue $\lambda_1 = \min\left\{ \|u\|_{\dot{H}^s}^2 : u \in \dot{H}^s(\Omega), \ \|u\|_{L^2} = 1 \right\}$, with $\lambda_1 = \lambda_1(\Omega) > 0$ and also corresponding eigenfunctions (which are positive and simple when $s \leq 1$; see, e.g., [7, Theorem 4.2]). Similarly, one has an increasing sequence of positive eigenvalues (repeated with multiplicities) going to infinity $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and corresponding eigenfunctions $v_1, v_2, \ldots$ giving an orthogonal base both of $L^2(\Omega)$ and of $\dot{H}^s(\Omega)$, so that $(-\Delta)^s v_k = \lambda_k v_k$ in $(\dot{H}^s(\Omega))'$ for any integer $k \geq 1$, i.e. $(v_k, u)_H = \lambda_k (v_k, u)_{L^2}$ for any $u \in \dot{H}^s(\Omega)$. Indeed it is enough to write $(u, v)_{L^2} = (Ku, v)_{\dot{H}^s}$ for some $K \in \mathcal{L}(\dot{H}^s)$ which is compact and self-adjoint and apply the spectral theorem.

For any fixed $\lambda \in \mathbb{R}$, consider the following nonlinear problem

$$(-\Delta)^s u - \lambda u - |u|^{2^*-2} u = 0 \quad \text{in } (\dot{H}^s(\Omega))', \tag{P$_\lambda$}$$

i.e. the Euler–Lagrange equation $d\mathcal{E}(u) = 0$ corresponding to the differentiable functional

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx. \tag{1.2}$$

It is worth noticing that when $\lambda < \lambda_1$, although the functional possesses the Mountain Pass geometry (arguing as in [18, Chapter II, Section 6]), the celebrated Mountain Pass lemma does not apply because the Palais–Smale condition fails. More generally, when $\lambda_k < \lambda < \lambda_{k+1}$ the functional has a linking geometry (using the spectral decomposition above and arguing again as in [18, Chapter II, Section 8]) but the usual minimax scheme still cannot be applied for the same reason. As it is well known when $s = 1$, this is due to the presence of a limiting nonlinearity in (1.2) and it is related to the lack of compactness for the associated critical Sobolev embedding $\dot{H}^s \hookrightarrow L^{2^*}$, which is a consequence of the invariance of the $\dot{H}^s$- and L^{2^*}-norms with respect to the scaling

$$u(\cdot) \rightsquigarrow \tilde{u}_{x_0,\eta}(\cdot) = \eta^{\frac{2s-N}{2}} u\left(\frac{\cdot - x_0}{\eta} \right), \tag{1.3}$$

for arbitrarily fixed $\eta > 0$ and $x_0 \in \mathbb{R}^N$.

In the seminal paper [3] the authors circumvent this difficulty proving that, for $s = 1$, a local (PS)-condition holds for $\lambda < \lambda_1$ small enough. Soon after a decisive breakthrough was obtained in [17], still in the local case $s = 1$, describing the precise mechanism responsible for the lack of the (PS)-condition; i.e., in Author's words, proving that *compactness for Palais–Smale sequences holds "apart from jumps of the topological type of admissible functions"*, a sense we will make precise below. This major advance paved the way to several extensions and to a huge number of applications, e.g. in the case of problems involving biharmonic and polyharmonic operators but also in other more complicated problems (see e.g. [18,19] and the references therein).

In order to state precisely our main result, consider the following limiting problem

$$(-\Delta)^s u - |u|^{2^*-2} u = 0 \quad \text{in } (\dot{H}^s(\Omega_0))', \tag{P$_0$}$$

where Ω_0 is either the whole $\mathbb{R}^N$ or a half-space; i.e. the Euler–Lagrange equation $d\mathcal{E}^*(u) = 0$ corresponding to the energy functional $\mathcal{E}^* : \dot{H}^s(\Omega_0) \to \mathbb{R}$,

$$\mathcal{E}^*(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx - \frac{1}{2^*} \int_{\Omega_0} |u|^{2^*} dx. \tag{1.4}$$

We have the following extension of the result in [17], describing Palais–Smale sequences for (1.2) in the full range $0 < s < N/2$,

[1] For further details on the fractional Sobolev spaces, we refer to [4] and the references therein.