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Stability of spherical caps under the volume-preserving mean curvature flow with line tension

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ABSTRACT

We show stability of spherical caps (SCs) lying on a flat surface, where the motion is governed by the volume-preserving Mean Curvature Flow (MCF). Moreover, we introduce a dynamic boundary condition that models a line tension effect on the boundary. The proof is based on the *generalized principle of linearized stability*.

1. Introduction

The geometric evolution law $V_\Gamma = H_\Gamma$, meaning that the motion of a point on the surface in normal direction V_Γ is equal to the mean curvature of the surface in that point, has many applications in geometry, physics and materials science. For example the evolution of grain boundaries is governed by mean curvature flow. First important results by mean curvature flow are due to Brakke [5], Gage and Hamilton [14] and Huisken [18]. The flow $V_\Gamma = H_\Gamma$ is known as the mean curvature flow (MCF) and with the additional condition of volume conservation, this flow appears e.g. as a model for surface attachment limited kinetics (SALK), see e.g. Cahn and Taylor [6]. In 1987 it was Huisken [19] and in 1998 Escher and Simonett [12], who provided important results concerning the volume-preserving MCF. Volume preserving mean curvature flow of rotationally symmetric surfaces with boundary contact has been studied by Athanassenas [2], see also the recent work [3]. Stability of cylinders under volume preserving mean curvature flow with a 90-degree angle condition at an external boundary has been studied by Hartley [16].

This paper is devoted to stability of spherical caps in $\mathbb{R}^3$ that lie on a flat surface $\mathbb{R}^2 \times \{0\}$. Modeling a drop of liquid or a soap bubble physics suggest that the air–liquid-interface, which can be viewed as an evolving hypersurface, tends to minimize its area. If such a surface gets into contact with some fixed impermeable boundary layer the mass conservation law makes it necessary to demand a constant volume condition. The occurring contact angle is mainly determined by the

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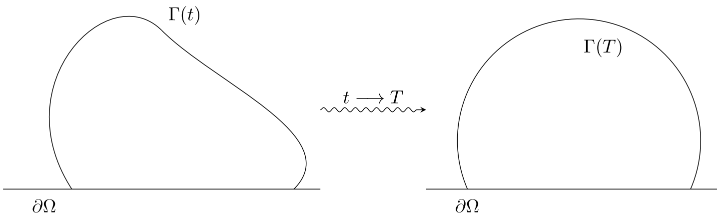

Fig. 1. Evolving hypersurface $\Gamma(t)$ in contact with a container boundary $\partial\Omega$.

material constants and thereby the wettability of the container. The free energy is given as

$$\mathcal{E}(\Gamma) := \int_\Gamma 1\, d\mathcal{H}^2 - \int_D a\, d\mathcal{H}^2$$

where $d\mathcal{H}^d$, $d \in \{1, 2\}$ denotes integration with respect to the d-dimensional Hausdorff measure, $a > 0$ is a constant and D is the wetted region. The first term measures surface energy and the second term accounts for contact energy. Then the angle α at the junction line is determined by $\cos\alpha = -a$, see Fig. 2 and Finn [13]. We remark that the contact angle, which is typically used in physics, is given as $\gamma = \pi - \alpha$. However, in particular on small length scales, a second effect is entering the scenery, namely the line tension (cf. Section 1 of [4]). This effect penalizes long contact curves and forces the drop or bubble to detach more from the boundary. The governing energy for a hypersurface $\Gamma \subseteq \mathbb{R}^3$ with contact to $\mathbb{R}^2 \times \{0\}$ is in this case given as

$$\mathcal{F}(\Gamma) := \int_\Gamma 1\, d\mathcal{H}^2 - \int_D a\, d\mathcal{H}^2 + \int_{\partial\Gamma} b\, d\mathcal{H}^1,$$

where $b > 0$ is a constant. The last term accounts for line energy effects. For a mathematical treatment of variational problems related to $\mathcal{F}$ we refer to Morgan [24,25], Morgan and Taylor [26] and Cook [7]. The motion of such an evolving hypersurface Γ, which is schematically illustrated in Fig. 1, will be a suitable gradient flow of the energy $\mathcal{F}$.

During this motion it seems artificial to prescribe the boundary curve or the contact angle, since an arbitrary drop or bubble, which is brought in contact with a solid container, will not instantly have a boundary curve or contact angle that is energetically minimal. Prescribing the contact curve or the contact angle would correspond to Dirichlet or Neumann boundary conditions, respectively. Instead of doing so, we will impose dynamic boundary conditions to allow the contact angle to change and the boundary curve to move. We will prove stability for spherical caps, which are the simplest stationary surfaces of the given flow. It will turn out that the set of equilibria forms a three-dimensional manifold. This is due to the fact that we have two degrees of freedom with respect to horizontal translations and another degree of freedom stems from a change in the enclosed volume. As a consequence the classical theory of linearized stability does not apply and we have to use the generalized principle of linearized stability as introduced by Prüss, Simonett and Zacher in [30].

After some elementary results on spherical caps in Section 2, we will introduce in Section 3 the generalized principle of linearized stability, which is the basis of our stability analysis. We will also introduce the abstract setting concerning the involved operators and spaces. Before we can apply the principle in Section 5 by checking the four assumptions that are needed and formulate our final stability result in Theorem 5.13, we need some perturbation result from semigroup theory to deal with the non-locality of the volume-preserving MCF in Section 4. In order to show stability of stationary solutions we in particular need to study the spectrum of the surface Laplacian on the spherical cap with non-standard boundary conditions.

2. Spherical caps

We want to consider the motion of an evolving hypersurface $\Gamma = (\Gamma(t))_{t\in I}$ inside the upper half space $\Omega := \mathbb{R}^3_+ := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, which remains in contact with the boundary $\partial\Omega$ given as the x-y-plane. With $U \subseteq \Omega$ we want to denote the region between Γ and $\partial\Omega$ and D shall be defined as $D := \partial U \cap \partial\Omega$. In particular, we have $\partial D = \partial\Gamma$. For a point $p \in \Gamma$ we denote the exterior normal to Γ in p by $n_\Gamma(p)$, where the term "exterior" should be understood with respect to U. Obviously, the normal n_D of U on D is the negative of the third unit vector. Furthermore, for a point $p \in \partial\Gamma$ we want to denote by $n_{\partial\Gamma}$ and $n_{\partial D}$ the outer conormals to $\partial\Gamma$ and ∂D in p. In addition, we define the tangent vector to the curve $\partial\Gamma$ by $\vec{\tau}(p) := \frac{c'(t)}{|c'(t)|}$ and its curvature vector by $\vec{\varkappa}(p) := \frac{1}{|c'(t)|}\left(\frac{c'(t)}{|c'(t)|}\right)'$, where $c : (t-\varepsilon, t+\varepsilon) \longrightarrow \partial\Gamma$ is a parametrization of $\partial\Gamma$ around $p \in \partial\Gamma$ with $c(t) = p$.

For two parameters $a \in \mathbb{R}$ and $b > 0$ the motion of Γ shall be driven by the volume-preserving mean curvature flow with a dynamic boundary condition

$$V_\Gamma(t) = H_\Gamma(t) - \overline{H}(t), \tag{2.1}$$
$$v_{\partial D}(t) = a + b\varkappa_{\partial D}(t) + \cos(\alpha(t)). \tag{2.2}$$