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Explosion of solutions to nonlinear stochastic wave equations with multiplicative noise[☆,☆☆]

Takeshi Taniguchi

Division of Mathematical Sciences, Graduate School of Comparative Culture, Kurume University, Miimachi, Kurume, Fukuoka 839-8502, Japan

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Takeshi Taniguchi dedicates the present paper to the memory of Professor Shi-geo Sasaki at Tohoku University

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ABSTRACT

Let $D \subset R^d$ be an open bounded domain in the d-dimensional Euclidian space R^d with smooth boundary ∂D. In this paper we investigate the local existence, global existence and explosion of solutions to the following stochastic wave equation:

$$\begin{cases} dX(t) = Y(t)dt, \\ dY(t) = (\varDelta X(t) - |Y(t)|^p Y(t) + |X(t)|^q X(t))dt \\ + B(t, X(t), Y(t))dW(t), \quad p, q > 0, \\ X(0) = X_0, Y(0) = Y_0, \ X(t) \mid \partial D = 0, \end{cases} \tag{0.1}$$

where $\varDelta = \sum \partial^2/\partial x_i^2$ is a Laplace operator. The process $W(t)$ denotes a cylindrical Brownian process on a complete probability space $(\varOmega, F, P)$ with filtration of the usual condition.

1. Introduction

The stochastic partial differential equations have been well investigated as the important models for investigating much phenomenon in real world, for example, like physical mechanics, economical science and biology and so on [1,6,10,16].

In this paper we investigate local and global existence, and explosion of solutions to the following stochastic wave equation:

$$\begin{cases} dX(t) = Y(t)dt, \\ dY(t) - \Delta X(t)dt + |Y(t)|^p Y(t)dt = |X(t)|^q X(t)dt + B(t, X(t), Y(t))dW(t), \\ X(0) = X_0, \qquad Y(0) = Y_0, \qquad X(t)\,|\partial D = 0, \quad p, q > 0, \end{cases} \tag{1.1}$$

where we assume that $D \subset R^d$ is an open bounded domain in R^d with smooth boundary ∂D. Let $(\varOmega, P, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a complete probability space with usual conditions. Let X_0 and Y_0 be $\mathcal{F}_0$-measurable.

As well known the deterministic wave equations have been extensively considered by many authors over the past three decades ([17,12] and references therein). One can find many references on the local and global existence, the asymptotic

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E-mail address: takeshi_taniguchi@kurume-u.ac.jp.

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behavior and explosion of solutions to the various type of deterministic nonlinear wave equations [11]. Among them Georgiev and Todorova [8] proved by the fixed point theorem the existence of local solutions to the following wave equation with the nonlinear damping term and source term:

$$u_{tt}(t) - \Delta u(t) + |u_t(t)|^p u_t(t) = |u(t)|^q u(t), \quad p, q > 0. \tag{1.2}$$

They proved that if the relation $p \geq q$ holds, then the solution exists globally for any initial value. They also considered a sufficient condition for a solution to blow up when $p < q$ holds. And then the sufficient conditions for the solution to blow up were discussed by many authors (e.g. Massaoudi [13], Vitillaro [18], Massaoudi and Said-Houari [14], Hu and Zhang [9]).

In this paper we will prove in Theorem 4 that a solution to (1.1) exists globally provided that $p \geq q$ and for any $z \in H_0^1(D)$ and $w \in L^{p+2}(D)$ there exist positive constants $\hat{\varepsilon}$ and c_β, c_γ such that

$$|B(t, z, w)|^2_{L_2^0} \leq |\sigma(t)|^2_{L_2^0} + \hat{\varepsilon}\, |w|^{p+2}_{p+2} + +c_\beta E\, |z|^{q+2}_{q+2} + c_\gamma \|z\|^2, \quad t \geq 0,$$

where the function $B(t, z, w)$ satisfies the local Lipschitz condition in variables z and w, and $\sigma(t)$ is a locally integrable function and $\hat{\varepsilon} > 0$ is small enough, and we also will prove that in Theorem 5 a solution to (1.1) blows up provided that $p < q$ and the initial value (X_0, Y_0) satisfies the following inequality:

$$H(0) := \frac{2}{q+2} E\, |X_0|^{q+2}_{q+2} - E\, |Y_0|^2_2 - E\, \|X_0\|^2 > 0, \tag{1.3}$$

the function $B(t, z, w)$ satisfies the local Lipschitz condition in variables z and w, and satisfies the following inequality: for $z \in H_0^1(D)$, $w \in L^{p+2}(D)$,

$$|B(t, z, w)|^2_{L_2^0} \leq |w|^{p+2}_{p+2} + |b(t, z)|^2_{L_2^0} + |\sigma(t)|^2_{L_2^0}, \quad t \geq 0, \tag{1.4}$$

where the function $b(\cdot, \cdot)$ satisfies that

$$2 \int_0^\infty E\left(|b(s, z)|^2_{L_2^0} + |\sigma(s)|^2_{L_2^0}\right) ds < H(0). \tag{1.5}$$

Furthermore, let

$$\frac{2}{q+2} E\, |X_0|^{q+2}_{q+2} - E\, |Y_0|^2_2 - E\, \|X_0\|^2 = 0.$$

If it holds that

$$|B(t, z, w)|^2_{L_2^0} \leq |w|^{p+2}_{p+2}, \quad w \in L^{p+2}(D), \quad t \geq 0, \tag{1.6}$$

then the same conclusion holds.

Here we note that stochastic nonlinear wave equations have been considered by many authors. Among them, Bo, Tang and Wang [2] investigated explosion of solutions to the following stochastic nonlinear wave equation with the linear damping term and the nonlinear source term:

$$u_{tt}(t) - \Delta u(t) + \gamma u_t(t) = |u(t)|^q u(t) + \delta g(t, x) dW(t), \quad \gamma, q > 0. \tag{1.7}$$

Recently H. Gao, F. Liang and B. Guo [7] considered the local existence, global existence and explosion of solutions to a stochastic nonlinear wave equation (1.1) perturbed by an additive noise $B(t, X(t), Y(t))dW(t) = \sigma(t, x)dW(t)$ (see Theorem 4.3, [7]). It is worthy saying in [5,3] Chow considered the explosion theorem of the solutions to stochastic nonlinear wave equation with multiplicative noise without a nonlinear damping term and Ondrejat [15] considered global mild solutions to stochastic wave equations.

But we note that unfortunately in their papers Bo, Tang and Wang [2], and H. Gao, F. Liang and B. Guo did not discuss the global existence and explosion of a solution to the stochastic wave equation with multiplicative noise which is more important and interesting, because of technical difficulty, that is a major hurdle (see 1350013-3, [7]).

The contents of this paper are as follows. In Section 2 we give preliminaries. In Section 3 the stochastic wave equation (3.1) with an additive noise is discussed. In Section 4 we prove the existence of a unique solution to (4.1) with a multiplicative noise under Conditions 1 and 2 by the Picard successive approximation. In Section 5 we discuss the existence of a local solution to (4.1) under Conditions 1 and 3. In Section 6 we consider the existence of a global solution to (1.1) under (5.1) and Conditions 1 and 4. In Section 7 we prove the main theorem, Theorem 5. In other words, a sufficient condition for the explosion of solutions to (1.1) is considered. In Section 8 an example which illustrates Theorem 5 is presented.

In this paper $c, c_* > 0$ and $C, C_* > 0$ denote constants which change from line to line. We often omit $\omega \in \Omega$ if no confusion arises. The paring $\langle u, u^* \rangle_{V,V^*}$ is also denoted by (u, u^*) by the same notation as the inner product of H.