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On critical points of the σ_2-energy over a space of measure preserving maps

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ABSTRACT

Let $\mathbf{X} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the σ_2-*energy* functional

$$\mathbb{F}_{\sigma_2}[u;\mathbf{X}] := \int_{\mathbf{X}} |\nabla u \wedge \nabla u|^2 \, dx,$$

over the space of *admissible* maps

$$\mathfrak{A}(\mathbf{X}) = \left\{ u \in W^{1,4}(\mathbf{X}, \mathbb{R}^n) : u|_{\partial\mathbf{X}} = x, \ \det \nabla u = 1 \text{ for } \mathcal{L}^n\text{-a.e. in } \mathbf{X} \right\}.$$

A good measure of how much a map u stretches areas (of 2-dimensional *sub*-manifolds of the domain $\mathbf{X}$) is the norm of $\nabla u \wedge \nabla u : \wedge^2 T\mathbf{X} \to \wedge^2 T\mathbf{X}$, analogously to $|\nabla u|^2$ (the *Dirichlet* energy density) that is a measure of length's stretching. These kinds of functionals also were arisen as a physical model describing the strong interactions of quantum fields which was introduced by T. Skyrme in 1961. In this paper we introduce a class of maps referred to as *generalised* twists and *examine* them in connection with the Euler–Lagrange equations associated with $\mathbb{F}_{\sigma_2}[\cdot;\mathbf{X}]$ over $\mathfrak{A}(\mathbf{X})$. In particular we present a novel characterisation of all *twist* solutions and this points at a surprising discrepancy between *even* and *odd* dimensions which follows very closely to the ideas that have been introduced by the second author in his recent paper Shahrokhi-Dehkordi and Taheri (2009) [17]. Indeed we show that in even dimensions the latter system of equations admits *infinitely* many smooth solutions, modulo isometries, amongst such maps. In odd dimensions this number reduces to *one*. The result relies on a careful analysis of the *full* versus the *restricted* Euler–Lagrange equations. We investigate various qualitative properties of these solutions in view of a remarkably interesting previously unknown *explicit* formula.

1. Introduction

Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be two compact Riemannian manifolds of dimensions m and n, respectively, and $u \in C^1(\mathcal{M}, \mathcal{N})$. The pullback u^*h is a *two*-covariant tensor field on $\mathcal{M}$. Assume $[\nabla u]^t : T\mathcal{N} \to T\mathcal{M}$ is the formal adjoint of ∇u with respect to g and h, then we can consider u^*h as an endomorphism

$$[\nabla u]^t \circ [\nabla u] : T\mathcal{M} \to T\mathcal{M},$$

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which is called the *Cauchy–Green* (strain) tensor of u, analogously to the case of deformations in nonlinear elasticity (*see*, e.g., [8]). In view of the Cauchy–Green tensor being *symmetric* and *positive* semidefinite, let $0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_m$ denoting the square root of its eigen-values. Recall that $(\lambda_i)_{i=1}^n$ are also called principal distortion coefficients of u.

The elementary symmetric functions in the eigen-values of u^*h represent a measure of the geometrical distortion induced by the map u. They are called *principal invariants* of ∇u and we denote them as follows

$$\sigma_1(u) := |\nabla u|^2 = \sum_{i=1}^m \lambda_i^2$$
$$\sigma_2(u) := \left|\wedge^2 \nabla u\right|^2 = \sum_{1\le i<j\le m} \lambda_i^2\lambda_j^2$$
$$\vdots$$
$$\sigma_m(u) := \left|\wedge^m \nabla u\right|^2 = \lambda_1^2\lambda_2^2\dots\lambda_m^2.$$

For a map $u : (\mathcal{M}, g) \to (\mathcal{N}, h)$ a good measure of how much it stretches volumes (of p-dimensional *sub*-manifolds of $\mathcal{M}$) is the norm of $\wedge^p \nabla u : \wedge^p T\mathcal{M} \to \wedge^p T\mathcal{N}$, analogously to $|\nabla u|^2$ (the *Dirichlet* energy density) that is a measure of lengths stretching. According to [9,11] the σ_p-*energy* is defined as the following functional

$$\mathbb{F}_{\sigma_p}[u; \mathcal{M}] := \frac{1}{2}\int_{\mathcal{M}} \sigma_p(u)\, \nu_g.$$

However the variational problem for the σ_p-energy has already been treated in [5,9,25], very little is known about its solutions [e.g., multiplicity *versus* uniqueness, existence of non-trivial *strong* local minimisers, *etc.*] and is not *fully* understood. From our point of view, the particularities of $p = 2$ case are worth to be outlined for their differential geometric and physical interest in its own and hopefully for providing hints for further investigations for higher σ_p-*energy* functionals.

The primary aim of this paper is to investigate *extremising* the σ_2-energy functional over the space of measure preserving maps when the domain and target manifolds are the same as a bounded Lipschitz domain $\mathbf{X}$ in $\mathbb{R}^n$. In view of the standard inner product on the domains in $\mathbb{R}^n$ the σ_2-energy functional can alternatively be represented in the following way

$$\begin{aligned}\mathbb{F}_{\sigma_2}[u; \mathbf{X}] &= \frac{1}{2}\int_{\mathbf{X}} \sigma_2(u)\, dx \\ &= \frac{1}{4}\int_{\mathbf{X}}\left[\left(\sum_{i=1}^n \lambda_i^2\right)^2 - \sum_{i=1}^n \lambda_i^4\right] dx \\ &= \frac{1}{4}\int_{\mathbf{X}}\left[|\nabla u|^4 - \left|[\nabla u][\nabla u]^t\right|^2\right] dx =: \frac{1}{4}\int_{\mathbf{X}} \mathbf{F}_{\sigma_2}(\nabla u)\, dx,\end{aligned}$$

where in the *last* implication we have used the fact that $(\lambda_i)_{i=1}^n$ are *singular*-values of the matrix ∇u. Motivated by the above representation in what follows we proceed by considering the σ_2-energy functional

$$\mathbb{F}_{\sigma_2}[u; \mathbf{X}] = \frac{1}{4}\int_{\mathbf{X}} \mathbf{F}_{\sigma_2}\big(\nabla u(x)\big)\, dx, \tag{1.1}$$

over the space of *admissible* maps

$$\mathfrak{A}(\mathbf{X}) = \left\{u \in W^{1,4}_\varphi(\mathbf{X}, \mathbb{R}^n) : \det \nabla u = 1 \text{ for } \mathscr{L}^n\text{-a.e. in } \mathbf{X}\right\}, \tag{1.2}$$

where

$$W^{1,4}_\varphi(\mathbf{X}) := \left\{u \in W^{1,4}(\mathbf{X}, \mathbb{R}^n) : u|_{\partial\mathbf{X}} = \varphi\right\},$$

and φ denotes the *identity* map while $\mathbf{F}_{\sigma_2}(\xi) = |\xi|^4 - |\xi\xi^t|^2$.

In this paper we are primarily concerned with the task of *extremising* the σ_2-energy functional, $\mathbb{F}_{\sigma_2}[\cdot; \mathbf{X}]$, over the space $\mathfrak{A}(\mathbf{X})$ and *examining* a special class of maps of *topological* significance as *solutions* to the associated system of Euler–Lagrange equations which can formally be written as

$$\begin{cases}\operatorname{div} \mathcal{T}[x, \nabla u(x)] = 0 & x \in \mathbf{X}, \\ \det \nabla u(x) = 1 & x \in \mathbf{X}, \\ u(x) = \varphi(x) & x \in \partial\mathbf{X}.\end{cases}$$

Note that the *divergence* operator acts *row*-wise and the tensor field $\mathcal{T}$ is defined through

$$\begin{aligned}\mathcal{T}[x, \xi] &= \mathbf{F}'_{\sigma_2}(\xi) - \mathfrak{p}(x)\xi^{-t} \\ &=: \mathfrak{T}[x, \xi]\xi^{-t},\end{aligned} \tag{1.3}$$