Nonlinear Analysis 117 (2015) 99–123

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Asymptotic behavior of a non-Newtonian flow in a thin domain with Navier law on a rough boundary

F.J. Suárez-Grau

Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, 41012-Sevilla, Spain

ARTICLE INFO

Article history:
Received 8 April 2014
Accepted 19 January 2015
Communicated by Enzo Mitidieri

MSC:
76A05
76A20
35B27
35Q30

Keywords:
Non-Newtonian flow
Thin fluid films
Rough boundary
Navier condition
Adherence condition
Asymptotic behavior

ABSTRACT

We consider a non-Newtonian flow in a thin domain of thickness ε. The flow is described by the 3D incompressible Navier–Stokes (Stokes) system with a nonlinear viscosity, being a power of the shear rate (power law) of flow index p.

The bottom of the domain is irregular by the presence of slight roughness of amplitude ε^δ and period ε^β, satisfying the relation $1 < \beta < \delta$. Assuming pure slip or partial slip with a friction coefficient $\varepsilon^{-\gamma}$, with $\gamma > 0$, on the rough boundary, we consider the limit when domain thickness tends to zero and we obtain different models depending on the magnitude δ with respect to $\frac{2p-1}{p}\beta - \frac{p-1}{p}$, and the magnitude γ with respect to $\frac{1}{p-1}$.

1. Introduction

Roughness of the solid surface as well as the rheological properties of the fluid has influence in the fluid–solid interface condition. For that reason, the choice of boundary conditions of fluid flow is a relevant problem to determine such influence. It is commonly accepted that viscous fluids adhere to rough surfaces, and so the no-slip condition at the rough surfaces of a domain is widely used. This does not seem always valid for the non-Newtonian fluids, indeed non-Newtonian fluids melt and solutions slip against the surface. This phenomenon has been related in many mechanical papers concerning non-Newtonian fluids (see [22,31]).

In this sense, it has been suggested that, in many cases, the velocity field of the fluid u_ε in a domain Ω_ε obeys a Navier condition at the rough surface $\Gamma_\varepsilon \subset \partial\Omega_\varepsilon$:

$$\left[S\,\nu\right]_\tau = -\lambda[u_\varepsilon]_\tau \quad \text{on } \Gamma_\varepsilon, \qquad u_\varepsilon\,\nu = 0, \quad \text{on } \Gamma_\varepsilon,$$

where S is the deviatoric viscous stress tensor, ν denotes the outside unitary normal vector to Ω_ε on Γ_ε, the subscript $_\tau$ denotes the orthogonal projection on the tangent space of Γ_ε, and $\lambda > 0$ is the friction coefficient.

Notice that depending on the value of λ, we shall consider either pure slip, partial slip or no-slip at the boundary. This means that the friction coefficient λ shall be either zero, positive or $+\infty$.

Recently, based on the so called rugosity effect, some results mathematically justify that viscous fluids adhere completely to the boundary of an impermeable domain. More precisely, these results account asymptotically for the transformation of

E-mail address: fjsgrau@us.es.

http://dx.doi.org/10.1016/j.na.2015.01.013

pure slip boundary conditions on a rough surface in no-slip boundary conditions, as the amplitude of the roughness vanishes, provided that the energy of the solutions is uniformly bounded and there is enough roughness of the oscillating boundaries. We refer to the pioneering paper [14] for the case of periodic and very smooth boundaries, and to [6,10–12] (see also [16,19,23,28]) for having a quite complete understanding for arbitrary boundaries.

In this paper, assuming Navier condition on a slightly rough surface of a thin domain, our goal is to identify, depending on the magnitudes of the amplitude of the roughness and the friction coefficient, the type of boundary condition in which the starting Navier condition is transformed in the limit.

Next, let us set the position of the problem. We consider a smooth bounded open set $\omega \subset \mathbb{R}^2$ and a function Ψ in $W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period $Z' = (-1/2, 1/2)^2$, satisfying

$$\mathrm{Span}(\{\nabla\Psi(z') : z' \in \mathbb{R}^2\}) = \mathbb{R}^2, \tag{1}$$

which always holds except in the case where Ψ is constant in one direction (i.e. not included roughness of riblets-type: $\Psi = \Psi(z_1)$ or $\Psi = \Psi(z_2)$).

Then, we consider a spatial domain Ω_ε determined through

$$\Omega_\varepsilon = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, \ -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right) < x_3 < \varepsilon\right\}, \tag{2}$$

where ε is a positive parameter devoted to tends to zero representing the characteristic thickness of the domain, and ε^δ and ε^β represent the amplitude and the period, respectively, of the roughness satisfying $1 < \beta < \delta < +\infty$, i.e. relation

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta-\beta} = 0, \qquad \lim_{\varepsilon\to 0} \varepsilon^{\beta-1} = 0. \tag{3}$$

By $\Gamma_\varepsilon \subset \partial\Omega_\varepsilon$ we denote the rough boundary of Ω_ε, that is

$$\Gamma_\varepsilon = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, \ x_3 = -\varepsilon^\delta \Psi\left(\frac{x'}{\varepsilon^\beta}\right)\right\}. \tag{4}$$

From the other side it is well-known that for non-Newtonian flow through a thin domain the non linear Poiseuille law is used. For that reason, in this study we deal with the case where the viscosity is not constant. Thus, we consider that the viscosity satisfies the non linear power law, which is widely used for melted polymers, oil, mud, etc. Denoting the shear rate by $\mathbb{D}[u_\varepsilon] = \frac{1}{2}(Du_\varepsilon + D^t u_\varepsilon)$, the viscosity as a function of the shear rate is given by

$$\eta_p(\mathbb{D}[u_\varepsilon]) = \mu|\mathbb{D}[u_\varepsilon]|^{p-2}, \quad p > 1,$$

where the two material parameters $\mu > 0$ and $1 < p < +\infty$ are called the consistency and the flow index, respectively. For simplicity we suppose $\mu = 1$.

Recall that $p = 2$ yields the Newtonian fluid. For $p < 2$ the fluid is pseudoplastic (shear thinning), which is the characteristic of high polymers, polymer solutions, and many suspensions, whereas for $p > 2$ the fluid is dilatant (shear thickening), whose behavior is reported for certain slurries, like mud, clay, or cement, and implies an increased resistance to flow with intensified shearing.

Therefore, assuming the fluid incompressible and in a stationary state, and depending on the value of p, the velocity u_ε and the pressure π_ε satisfy:

- For $9/5 \le p < +\infty$, the non-Newtonian Navier–Stokes system

$$\begin{aligned} &-\mathrm{div}\left(\eta_p(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]\right) + (u_\varepsilon\nabla)u_\varepsilon + \nabla\pi_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon,\\ &\mathrm{div}\, u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon. \end{aligned} \tag{5}$$

- For $1 < p < 9/5$, due to the known technical difficulties with inertial term for $p < 9/5$, the non-Newtonian Stokes system

$$\begin{aligned} &-\mathrm{div}\left(\eta_p(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]\right) + \nabla\pi_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon,\\ &\mathrm{div}\, u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \end{aligned} \tag{6}$$

where the right-hand side f_ε is of the form

$$f_\varepsilon(x) = \widetilde{f}\left(x', \frac{x_3}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega_\varepsilon, \tag{7}$$

with $\widetilde{f}$ assumed in $L^{p'}(\omega \times (-1, 1))^3$.

In order to study the roughness effects on the flow, we assume Navier condition on Γ_ε,

$$\Big[\eta_p(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]\,\nu\Big]_\tau = -\lambda\,\varepsilon^{-\gamma}\big[u_\varepsilon\big]_\tau \quad \text{on } \Gamma_\varepsilon, \qquad u_\varepsilon\,\nu = 0 \quad \text{on } \Gamma_\varepsilon, \tag{8}$$