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A sharp lower bound for the first eigenvalue on Finsler manifolds with nonnegative weighted Ricci curvature

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ABSTRACT

Let (M, F) be an n-dimensional compact Finsler manifold without boundary or with a convex boundary and λ_1 be the first (nonzero) closed or Neumann eigenvalue of the Finsler Laplacian on M with nonnegative weighted Ricci curvature. In this paper, we prove that $\lambda_1 \geq \frac{\pi^2}{d^2}$, where d is the diameter of M, and that the equality holds if and only if M is a 1-dimensional circle or a 1-dimensional segment, which generalize the well-known Zhong–Yang's sharp estimate in Riemannian geometry (Zhong and Yang, 1984).

1. Introduction

The study of the lower bound of the first (nonzero) eigenvalue of Laplacian on Riemannian manifolds has a long history. For example, see Lichnerowicz [6], Obata [8], Cheeger [2], Li–Yau [5], and so on. For an overview, the reader is referred to the introduction of [1] and Chapter 3 in book [12], and references therein.

As a generalization of Riemannian geometry, Finsler geometry has been received more and more attentions recently since it has more and more applications in natural science. A Finsler manifold (M^n, F) means an n-dimensional smooth differential manifold equipped with a Finsler metric $F : TM \setminus \{0\} \to [0; +\infty)$ (see details in Section 2). On a Finsler manifold (M, F), the Laplace operator (often called the *Finsler Laplacian*) was introduced by Z. Shen via a variation of the energy functional (cf. [13,14]). If F is Riemannian, then the Finsler Laplacian is exactly the usual Laplacian. Unlike the usual Laplacian, the Finsler Laplacian is a nonlinear elliptic operator. The standard linear elliptic theory cannot be directly applied to the Finsler Laplacian. In spite of this, some progress has been made on the global analysis on Finsler manifolds in recent years [3,10,11,19,21]. In particular, inspired by the method of the gradient comparison developed in [1], G. Wang and C. Xia gave a sharp lower bound for the first eigenvalue of the Finsler Laplacian on a compact Riemannian manifold without boundary or with convex boundary, whose weighted Ricci curvature Ric_N is bounded below from a real number K, where $N \in [n, \infty]$.

Based on Wang–Xia's result (see Proposition 3.1 in Section 3), we can get a unified lower bound for the first closed or Neumann eigenvalue of the Finsler Laplacian if $\mathrm{Ricci}_N \geq K$ for some real numbers $N \in [n, \infty]$ and $K \in \mathbb{R}$. Explicitly, we have

Theorem 1.1. *Let (M^n, F, m) be an n-dimensional compact Finsler manifold, equipped with a Finsler structure F and a smooth measure m, without boundary or with a convex boundary. Assume that $\mathrm{Ric}_N \geq K$ for some real numbers $N \in [n, \infty]$ and $K \in \mathbb{R}$.*

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Let λ_1 be the first (nonzero) closed or Neumann eigenvalue for the Finsler Laplacian, i.e.,

$$\Delta_m u = -\lambda_1 u, \quad \text{in } M \tag{1.1}$$

with the Neumann boundary condition

$$\nabla u \in T_x(\partial M), \tag{1.2}$$

if ∂M is not empty. Then

$$\lambda_1 \geq \sup_{s\in(0,1)} \left\{4s(1-s)\frac{\pi^2}{d^2} + sK\right\}, \tag{1.3}$$

where d is the diameter of M.

The precise definition of the Finsler manifold, convex boundary, diameter, weighted Ricci curvature Ric_N, Finsler-Laplacian and the first eigenvalue, etc. will be given in Section 2. Note that $d \leq \pi\sqrt{\frac{N-1}{K}}$ if $K > 0$ by Theorem 7.3 in [9]. By a direct calculation, it is easy to see that

$$\sup_{s\in(0,1)} \left\{4s(1-s)\frac{\pi^2}{d^2} + sK\right\} = \begin{cases} 0, & \text{if } Kd^2 < -4\pi^2, \\ \left(\dfrac{\pi}{d} + \dfrac{Kd}{4\pi}\right)^2, & \text{if } Kd^2 \in [-4\pi^2, 4\pi^2], \\ K, & \text{if } Kd^2 \in \left(4\pi^2, \dfrac{N-1}{K}\pi^2\right]. \end{cases} \tag{1.4}$$

In particular, if $K = 0$, then the right side in (1.3) arrives the maximum $\frac{\pi^2}{d^2}$. In this case, $\lambda_1 \geq \frac{\pi^2}{d^2}$, which is optimal in the sense of Theorem 1.2. If F is Riemannian, then Theorem 1.1 is reduced to Theorem 1.1 in [15].

Theorem 1.2. *Let (M^n, F, m), λ_1 and d be the same as in Theorem 1.1. Assume the weighted Ricci curvature Ric_N of M is nonnegative for $N \in [n, \infty]$. Then $\lambda_1 \geq \frac{\pi^2}{d^2}$ and the equality holds if and only if M is a 1-dimensional segment or 1-dimensional circle.*

In particular, if (M^n, F) is a Riemannian manifold without boundary or with a convex boundary, then Theorem 1.2 is reduced to Zhong–Yang and Hang–Wang's results, which asserted that $\lambda_1(M^n) \geq \frac{\pi^2}{d^2}$ if M has nonnegative Ricci curvature and the equality holds if and only if M is a 1-dimensional circle or 1-dimensional segment [4,22]. In Finslerian case, the authors considered the optimal lower bound for the first eigenvalue on compact Finsler manifolds with nonnegative ∞-weighted Ricci curvature Ricci_∞ under some extra assumptions in [20]. Thus, Theorem 1.2 extends Zhong–Yang's well known sharp estimate in Riemannian case and Yin–He–Shen's result in Finslerian case. It is worth mentioning that the proof of Theorem 1.2 is not based on the gradient estimate of the eigenfunctions, which was used in [20,22], but on a comparison theorem on the gradient of the first eigenfunction with that of a one dimensional model function, which was given in [19].

This paper is organized as follows. In Section 2, we briefly review the necessary preliminaries on Finsler geometry. In Section 3, we prove Theorem 1.1. Moreover, as a special case of Theorem 1.1, one obtains that the lower bound estimate for the first eigenvalue on Finsler manifolds with nonnegative weighted Ricci curvature. We will give another proof of this estimate and meanwhile determine the necessary condition for which the equality holds (see Theorem 3.1). Finally, we characterize the equality in Theorem 3.1 and prove Theorem 1.2.

2. Finsler geometry

In this section, we briefly recall some fundamental concepts in Finsler geometry. For more details, we refer to [9–11,13,19], etc.

2.1. Finsler metric

Let M be an n-dimensional smooth manifold. A *Finsler structure* (or *Finsler metric F*) on M means a function $F : TM \to [0, \infty)$ with the following properties: (1) F is C^∞ on $TM \setminus \{0\}$; (2) $F(x, \lambda y) = \lambda F(x, y)$ for any $(x, y) \in TM$ and all $\lambda > 0$; (3) the matrix $(g_{ij}(x, y)) = (\frac{\partial^2 F(x,y)}{\partial y^i \partial y^j})$ is positive. Such a pair (M, F) is called a *Finsler manifold*. A Finsler structure F is said to be *reversible* if $F(x, -y) = F(x, y)$. Otherwise, F is non-reversible.

For $x_1, x_2 \in M$, the *distance function* from x_1 to x_2 is defined by

$$d(x_1, x_2) := \inf_\gamma \int_0^1 F(\dot\gamma(t))dt,$$

where the infimum is taken over all C^1 curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Note that the distance function may not be symmetric unless F is reversible. The *diameter* of M is defined by

$$d := \sup_{x,y\in M} d(x, y).$$