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Some theoretical results concerning diphasic flows in thin films

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ABSTRACT

We are interested in a model for diphasic fluids in thin flows taking into account both the hydrodynamical and the chemical effects at the interface between the two fluids. A limit problem in thin curved channels is introduced heuristically. It is a system coupling the Reynolds equation and the Cahn–Hilliard equation. We study the mathematical properties of this system, and prove an existence result under some smallness condition on the data.

1. Introduction

In many applications, the geometry of the flow is anisotropic (i.e. one dimension is small with respect to the others), e.g. in lubrication problems. In the Newtonian case, the flow of a fluid between two close surfaces in relative motion is described by an asymptotic approximation of the Navier–Stokes equations, the Reynolds equation. This equation makes it possible to uncouple the pressure and the velocity. Indeed, in thin films, the pressure is considered to be independent of the direction in which the domain is thin. Thus an equation on the pressure only is obtained, and the velocity can be deduced from the pressure. This approach was introduced by Reynolds, and has been rigorously justified in [3] for the Stokes equation, and generalized afterwards in many works: for the steady-case Navier–Stokes equations [2], for the unsteady case [4], for compressible fluids with the perfect gases law [20]. It is of interest to investigate how this approach can be used for the case of a two fluid flow.

A first diphasic model consists in introducing a variable viscosity η, which is either equal to the viscosity η_1 of one fluid or the viscosity η_2 of the other fluid (that is to say that the fluids are considered to be non-miscible). The behavior of η is described by a transport equation. In that case, when assuming the interface between the two fluids to be the graph of a function, the asymptotic equations corresponding to the thin film approximation can be interpreted as a generalized Buckley–Leverett equation, which governs the behavior of the saturation (i.e. the proportion of one fluid in the mixture) inside the gap, coupled with a generalized Reynolds equation, which governs the behavior of the pressure. These equations are investigated in [22] without shear effects, and in [5], [12] with shear effects. One of the main disadvantages of the method is that the fluid interface is supposed to be the graph of a function, which hinders for example the formation of bubbles. In addition, this kind of model only takes into account hydrodynamical effects between the two phases, and surface tension effects are neglected.

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The second class of models describing diphasic flows, which has been used up to now only for the Navier–Stokes equations, is the class of the so-called diffuse interface models. They take into account chemical properties at the interface between the two fluids, enabling an exchange between the two phases. In this paper, we use a Cahn–Hilliard equation, which involves an interaction potential, enhanced with a transport term. Thus this model describes both the chemical and the hydrodynamical properties of the flow. An order parameter φ is introduced, for example the volumic fraction of one phase in the mixture. The surface tension can be taken into account *via* an additional term depending on φ in the Navier–Stokes equations. This kind of model has been studied for the complete Navier–Stokes equations in [6], and for viscoelastic fluids in [10].

In this paper, we consider an asymptotic system (i.e. a thin film approximation) for a diphasic fluid modeled by the Cahn–Hilliard equation. In a similar way as for the Newtonian case, the Navier–Stokes equations are approximated by a modified Reynolds equation, in which the viscosity is not constant anymore. We study the Reynolds/Cahn–Hilliard system, and prove the existence and the regularity of a weak solution under a smallness assumption on the initial data and the geometry.

Let us describe briefly the main steps of the mathematical analysis. First, we study the Reynolds equation and investigate the regularity of the pressure and the velocity as functions of the order parameter. Next, we prove the existence of a solution to the system Reynolds/Cahn–Hilliard, by using a Galerkin process, which consists in introducing finite dimension approximations of φ. After obtaining *a priori* estimates for these approximations, we conclude that they converge to a solution of the system Reynolds/Cahn–Hilliard.

This paper is organized as follows. In Section 2, we introduce the two-dimensional model for a diphasic fluid in a thin film, which consists of a generalized Reynolds equation and of a diffuse-interface model (the Cahn–Hilliard equation). In Section 3, we state the main theorem, and give the main steps and difficulties of the proof. In Section 4, we deal with the Reynolds equation, and obtain some existence and regularity result on the velocity field and the pressure. In Section 5, we first introduce some specific results on trace estimates and Poincaré inequalities. They are used in the rest of the section for obtaining *a priori* estimates for the Cahn–Hilliard equation. At last, convergence results are deduced from these estimates, and allow to conclude the proof of the main theorem. Section 6 presents some preliminary numerical results obtained with this model in order to highlight the features of the model.

2. Modeling a diphasic fluid in a thin film

In this section, we will first present how a fluid is described in a thin domain by the Reynolds equation. Next, we introduce the hydrodynamical Cahn–Hilliard model for any fluid. Lastly, we combine both aspects and state the model of a diphasic fluid in a thin domain.

We introduce the physical domain $\bar{\Omega}$

$$\bar{\Omega} = \left\{(\bar{x}, \bar{z}) \in \mathbb{R}^2,\ 0 < \bar{x} < L,\ 0 < \bar{z} < h(x)\right\}. \tag{1}$$

The thin film approximation for an incompressible fluid leads to the following equations (see [3]), describing the behavior of the pressure p and the velocity field $\boldsymbol{u} = (u, v)$, η being the viscosity of the fluid.

$$\partial_{\bar{z}}\,(\eta\,\partial_{\bar{z}} u) = \partial_{\bar{x}} p, \qquad \partial_{\bar{z}} p = 0, \qquad \partial_{\bar{x}} u + \partial_{\bar{z}} v = 0.$$

In these equations, the thin film assumption leads to the decoupling of the pressure and the velocity, as well as the simplification of the equations.

We will see that it is possible to prove an existence theorem assuming a small size condition on the physical domain $\bar{\Omega}$ (see Theorem 3.3). In order to understand the dependence of the solution with respect to the domain $\bar{\Omega}$, we rescale the spatial variable $(\bar{x}, \bar{z})$ using a dilatation coefficient λ. More precisely, we suppose that the domain is small and can be written as

$$\bar{\Omega} = \{(\bar{x}, \bar{z}) \in \mathbb{R}^2, 0 < \lambda\bar{x} < \lambda L, 0 < \lambda\bar{z} < \lambda h(x)\},$$

and we rewrite the system using the following change of variable and domain

$$\lambda\bar{x} \to x, \qquad \lambda\bar{z} \to z, \qquad \bar{\Omega} \to \Omega = \{(x, z) \in \mathbb{R}^2, 0 < x < L, 0 < z < h(x)\}. \tag{2}$$

We assume that there exists three constants $(h_m, h_M, h'_M) \in \mathbb{R}^3_+$ such that the function $h \in \mathcal{C}^2(\mathbb{R})$ (see Fig. 1) satisfies

$$\forall x \in [0, L], \quad 0 < h_m \leqslant h(x) \leqslant h_M \quad \text{and} \quad |h'(x)| \leqslant h'_M, \tag{3}$$

and $h'(L) = 0$ as well as

$$\exists\tilde{\varepsilon} > 0 \quad \text{such that } \forall x \in [0, \tilde{\varepsilon}], \qquad h'(x) = h''(x) = 0.$$

Observe that the regularity of h ensures that the domain Ω defined by (2) satisfies the segment property and cone property (see [1, Section 4.2 and 4.3]).

The Reynolds equation now writes

$$\partial_z\,(\eta\,\partial_z u) = \lambda\,\partial_x p, \qquad \partial_z p = 0, \qquad \partial_x u + \partial_z v = 0. \tag{4}$$