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Morse indices of solutions for super-linear elliptic PDE's

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ABSTRACT

In this paper, we establish L^∞ estimates for solutions of some general superlinear and subcritical elliptic equations via the Morse index. Our results generalize and improve the previous works (Bahri and Lions, 1992; Yang, 1998).

1. Introduction

Consider the following elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a smooth bounded domain and f is a $C^1(\Omega \times \mathbb{R})$ function that we will specify later. Let u be a classical solution, define

$$\Lambda_u(h) := \int_\Omega |\nabla h|^2 - \int_\Omega f'(x,u)h^2, \quad \forall\, h \in C_c^1(\Omega). \tag{1.2}$$

Here $f'(x,t) := \frac{\partial f}{\partial t}(x,t)$. The Morse index of u, $i(u)$, is defined as the maximal dimension of all subspaces X of $C_c^1(\Omega)$ such that $\Lambda_u(h) < 0$ for any $h \in X \setminus \{0\}$. A solution u is said stable if $i(u) = 0$.

By using the symmetric Mountain Pass theorem, Ambrosetti and Rabinowitz in [1] showed that when f is super-linear, odd in u and has a subcritical growth, Eq. (1.1) has infinitely many solutions u_k such that $\lim_{k\to\infty} i(u_k) = \lim_{k\to\infty} \|u_k\|_{L^\infty} = \infty$ (see also [5,20] for more general results). In the celebrated paper [2], Bahri and Lions studied the perturbed equation i.e. $f(x,u) := |u|^{p-2}u + g(x,u)$ where $2 < p < \frac{2N-2}{N-2}$ and $g(x,u)$ is not assumed to be odd in u. They proved the existence of infinitely many solutions under appropriate growth restriction on g. This result was improved by Ramos, Tavares and Zou [18], using the Morse index of solutions for unperturbed problem, they obtained a sequence of sign-changing solutions to a more

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general problem. In [9], de Figueiredo and Yang also considered problem (1.1) where the associated Euler–Lagrange functional does not satisfy the Palais–Smale condition. Variational and blow-up techniques were applied together with Morse's index to derive existence results (see also [17,23]).

The idea of using the Morse index of solutions to obtain further qualitative properties of solutions to a semilinear elliptic equations was first explored in the subcritical case by Bahri and Lions in [3]. They proved

Theorem 1.1 *([3]). Assume that f satisfies:*

$$f'(x,s)|s|^{-p+1} \to c(x) > 0 \quad \text{uniformly on } \Omega, \text{ as } s \to \pm\infty,$$

where $c \in C(\overline{\Omega})$ and $1 < p < \frac{N+2}{N-2}$, then for any sequence of solutions u_n to (1.1),

$$i(u_n) \to \infty \quad \text{if and only if } \|u_n\|_{L^\infty(\Omega)} \to \infty.$$

Suppose that $\|u_n\|_{L^\infty} \to \infty$ while $i(u_n)$ remains bounded, by blow-up technique, Bahri and Lions obtained a nontrivial bounded solution having finite Morse indices in the whole space or in the half space with Dirichlet conditions for the Lane–Emden equation:

$$-\Delta u = |u|^{p-1}u. \tag{1.3}$$

On the other hand, a spectral argument combined with the Pohozaev identity showed that $u \equiv 0$, hence the contradiction.

In [15], the authors extended Theorem 1.1 when f does not have the same asymptotic behavior at $+\infty$ and $-\infty$, namely f satisfies the following assumption

$$(H)\, f'(x,t) \sim p^+ t^{p^+-1} \quad \text{at } +\infty, \qquad f'(x,t) \sim p^-|t|^{p^--1} \quad \text{at } -\infty, \text{ uniformly in } x,$$

with $p^- \neq p^+$ satisfying $p^-, p^+ \in \left(1, \frac{N+2}{N-2}\right)$, if $N \geq 4$ and $p^-, p^+ \in (2,5)$ if $N = 3$. The proof in [15] is harder than [3], since the "blow-up" argument leads to deal with $-\Delta u = u_+^p$. They had to use the classification result in [14] together with Harnack's inequality and barrier functions estimates.

In the supercritical case, Theorem 1.1 was first extended by Dancer [7] with the restriction $\frac{N+2}{N-2} < p < \frac{N}{N-4}$ if $N \geq 4$. In an elegant paper, Farina [11] obtained a sharp classification for all finite Morse indices solutions of (1.3) (see also [10]). This classification is useful to prove Theorem 1.1 when $1 < p < p_c(N)$, where $p_c(N)$ is the so-called Joseph–Lundgren exponent, which is much bigger than $\frac{N+2}{N-2}$. After that, using Harnack's inequality and combining with similar L^p-estimates derived in [11], Rebhi extended the result of [15] up to the optimal exponent $p_c(N)$ [19]. In [8], Davilla, Dupaigne and Farina considered Eq. (1.3) in a general open set $\Omega \subset \mathbb{R}^N$, without any boundary conditions. By a boot-strap technique, the authors proved some regularity results for weak solutions in $H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ with finite Morse index, and they provided a universal estimate for classical solutions of $-\Delta u = f(u)$ in Ω, where f has an asymptotical behavior like $|s|^{p-1}s$ at infinity.

Motivated by [3], based on local interior estimates and careful boundary estimates, Yang obtained in [22] some explicit estimates of L^p or L^∞ norm for solutions to (1.1) via their Morse index. In particular, under weaker conditions than [3], Yang controlled the L^p or L^∞ norm of solution by polynomial growth in Morse index. More precisely, consider the following conditions:

(H_1) (Super-linearity) There exists $\mu > 0$ such that

$$f'(x,s)s^2 \geq (1+\mu)f(x,s)s > 0, \quad \text{if } |s| > s_0,\ x \in \Omega.$$

(H_2) (Subcritical growth) There exists $0 < \theta < 1$ such that

$$\frac{2N}{N-2}F(x,s) \geq (1+\theta)f(x,s)s, \quad \text{if } |s| > s_0,\ x \in \Omega,$$

where $F(x,s) = \int_0^t f(x,t)dt$.

(H_3) There is a constant $C \geq 0$ such that

$$|\nabla_x F(x,s)| \leq C(F(x,s)+1), \quad x \in \Omega.$$

Yang proved then

Theorem 1.2 *(Theorems 2.1–2.2 in [22]). If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of (1.1) and f satisfies (H_1), (H_2) and (H_3), then there exists a positive constant $C = C(\Omega, f)$ such that*

- *$\int_\Omega |f(x,u)|^{p_0} \leq C(i(u)+1)^\alpha$ where $p_0 = 1 + \frac{(1+\theta)(N-2)}{(1-\theta)N+2(1+\theta)}$ and $\alpha = \left(\frac{3}{2} + \frac{3}{2+\mu}\right)\frac{(2+\mu)^2}{3\mu+\mu^2}$.*
- *There exists a constant β satisfying $0 < \beta \leq \frac{2\alpha}{p_0 N(2-p_0)}\left[\frac{2}{N(2-p_0)} - \frac{1}{p_0}\right]^{-1}$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^\beta.$$

The results in [15] were generalized in [13] to similar equation with the Neumann boundary conditions. However, in that case, the techniques used in [22] cannot be adapted, due to the boundary estimates.