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Singularly perturbed elliptic problems with nonautonomous asymptotically linear nonlinearities

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ABSTRACT

We consider a class of singularly perturbed elliptic problems with nonautonomous asymptotically linear nonlinearities. The dependence on the spatial coordinates comes from the presence of a potential and of a function representing a saturation effect. We investigate the existence of nontrivial nonnegative solutions concentrating around local minima of both the potential and of the saturation function. Necessary conditions to locate the possible concentration points are also given.

1. Introduction

In this paper we study the existence of positive solutions of the problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u^2 = \dfrac{u^3}{1+s(x)u^2} & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_ε}$$

for $N \geq 2$, $\varepsilon > 0$ a small parameter and $V,\ s : \mathbb{R}^N \to \mathbb{R}$ Hölder continuous functions such that

$$s(x) \geq \alpha > 0 \quad \forall x \in \mathbb{R}^N, \tag{1.1}$$

$$V(x) \geq \mu > 0, \quad \forall x \in \mathbb{R}^N. \tag{1.2}$$

It is well known that every positive solutions u_ε of (P_ε) generates a standing wave, i.e. $\phi_\varepsilon(x, t) = u_\varepsilon(x)e^{-iEt/\hbar}$ solution of

$$i\hbar\partial_t \phi + \frac{\hbar^2}{2m}\Delta\phi - W(x)\phi = \frac{\phi^3}{1+s(x)\phi^2} \tag{1.3}$$

for $W = V + E$, $\varepsilon^2 = \hbar/2m$.

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Problem (1.3) represents the propagation of a light pulse along a saturable medium. A typical class of saturable medium is constituted by the photorefractive crystals, one of the most preferable materials to observe the propagation of a light beam, because of their slow response to the propagation, making easier the observation. When a beam passes through these materials its refractive index changes so that the light remains confined and solitons are generated. When observing light propagation through these media one can see a *saturation effect*: it is possible to increase the amplitude of the generated solitons by increasing light intensity up to a critical bound characteristic of the material. This kind of interaction is not well represented by the usual Schrödinger equation with cubic nonlinearity, so that this model is replaced by (1.3) where the usual autointeraction represented by the cubic power is prevalent for "small" u, while a linear interaction, $u/s(x)$, is predominant for "large" u. Moreover, aiming to analyze the observation through different materials we admit a possible change of the saturation feature in dependence on the spatial coordinates, which may happen observing the propagation along different material.

An interesting and largely studied class of solutions of (P_ε) is the family of semiclassical states, that are families u_ε with a spike shape concentrating around some points of $\mathbb{R}^N$ for ε sufficiently small. There is a broad variety of contributions concerning the existence of this kind of solutions for the equation

$$-\varepsilon^2\Delta u + V(x)u = f(x,u). \tag{1.4}$$

For $f(x,t) = t^3$, the first contribution on the subject in the one dimensional case is due to Floer and Weinstein [12] who show the existence of a solution u_ε concentrating around any given x_0 nondegenerate critical point of $V(x)$. Their result has been extended in higher dimension in [20,21] for $f(x,t) = |t|^{p-1}t$ with $1 < p < (N+2)/(N-2)$. The common approach used in these papers is a Lyapunov–Schmidt reduction, consisting in a local bifurcation type result, which relies on the uniqueness and nondegeneracy of the ground state solution of the autonomous problem

$$-\Delta v + V(x_0)v = f(x_0, v). \tag{1.5}$$

The Lyapunov–Schmidt procedure or more general finite-dimensional reductions methods have been used to find solutions concentrating around any x_0 isolated minimum (or maximum) point with possibly polynomial degeneration of V in [1], and then around stable critical points in [2,13,16,22] (see also the interesting books [3,8] and the references therein).

A different approach to this is to find a solution u_ε for ε positive and then study its asymptotic behavior for ε tending to zero. This procedure has been firstly used by Rabinowitz in [24] assuming that $\inf V(x) < \liminf_{|x|\to+\infty} V(x)$ and proving concentration around a local minimum point of V. This philosophy has been improved in [10,11], where it is shown, by means of a penalization argument, that it is sufficient to assure a local condition on the potential: there exists a bounded open set Λ such that

$$\inf_\Lambda V < \inf_{\partial\Lambda} V.$$

As for the reduction method also this procedure has been used to extend the existence and concentration result in many different directions (see [11,6,7,5]).

When passing in (1.5) from $f(t) = k(x)t^3$ to $f(x,t) = t^3/(1+s(x)t^2)$ many differences arise. First of all, thanks to (1.1), we do not have a critical exponent as $|f(t)| < t/\alpha$. Moreover, as f is asymptotically linear, the action functional I_ε, defined in

$$\mathbb{H}^1 = \left\{u \in H^1(\mathbb{R}^N) : V(x)u^2 \in L^1(\mathbb{R}^N)\right\}, \tag{1.6}$$

by

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N}\left[\varepsilon^2|\nabla u(x)|^2dx + V(x)u^2(x)\right]dx - \int_{\mathbb{R}^N} F(x,u(x))dx,$$

for $F(x,t)$ given by

$$F(x,t) = \frac{1}{2s(x)}t^2 - \frac{1}{2s^2(x)}\ln(1+s(x)t^2), \tag{1.7}$$

may present different geometric behavior in dependence of V and s, e.g. if $V(x)s(x) > 1$ for every $x \in \mathbb{R}^N$, I_ε is always positive, convex and has only a global minimum at $u \equiv 0$. For V and s constant and such that $Vs < 1$ in [28] it is proved the existence of a positive radially symmetric solution which is showed to be unique according to [26,27]. Regarding the existence of semiclassical states, in [14] it is studied this kind of problem for general autonomous nonlinearity $f(x,t) = f(t)$, asymptotically linear or not, and it is shown the existence of a positive solution u_ε concentrating around a local minimum of V via variational methods and penalization arguments. Here, being interested in the possible interaction between $V(x)$ and $s(x)$, we will deal with the following autonomous, or frozen, problem

$$\begin{cases} -\Delta u + V(y)u = \dfrac{u^3}{1+s(y)u^2} & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{S_y}$$