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On the Cauchy problem for a two-component b-family system

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ABSTRACT

This paper is concerned with the Cauchy problem for a two-component b-family system. We prove that the solution map of the Cauchy problem of the b-family system is not uniformly continuous in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 5/2$.

1. Introduction

In this paper, we consider the following nonlinear dispersive equation

$$\begin{cases} m_t + um_x = k_1 mu_x + k_2 \rho\rho_x, & t > 0,\ x \in \mathbb{R}, \\ \rho_t = k_3(\rho u)_x, & t > 0,\ x \in \mathbb{R}, \\ m(0, x) = m_0(x), \qquad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$. As stated in [1], there are two cases about this system: (i) $k_1 = b,\ k_2 = 2b$ and $k_3 = 1$; (ii) $k_1 = b + 1$, $k_2 = 2$ and $k_3 = b$ with $b \in \mathbb{R}$. System (1.1) was recently introduced by Guha [2]. Zou [3] considered the system (1.1) and obtained some properties of the solutions.

If $\rho \equiv 0$, system (1.1) will become the b-family equation

$$u_t - u_{txx} + c_0 u_x + (b + 1)uu_x + \rho u_{xxx} = bu_x + uu_{xxx}. \tag{1.2}$$

For any $b \neq -1$, (1.2) can be derived as the family of asymptotically equivalent shallow water wave equation that emerge at quadratic order accuracy by an appropriate Kodama transformation, see [4,5] for the details. For $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated, see [4,5].

When $b = 2$ and $\rho = 0$, Eq. (1.2) becomes the Camassa–Holm equation, modelling the unidirectional propagation of shallow water waves over a flat bottom. The Cauchy problem of the Camassa–Holm equation [6] has been extensively

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considered, see [7–10]. For $b = 3$ and $c_0 = \rho = 0$, Eq. (1.2) becomes the Degasperis–Procesi equation [11], which is regarded as a model for nonlinear shallow water dynamics. There are also many papers involving the Degasperis–Procesi equation, cf. [12,1]. In paper [13], Constantin–Lannes studied the relevance of the Camassa–Holm and Degasperis–Procesi equations.

For $\rho \not\equiv 0$ and if $k_1 = 2$, the system (1.1) becomes the two-component Camassa–Holm system [14,15]

$$\begin{cases} m_t + um_x + 2mu_x + \sigma\rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$. Local well-posedness of system (1.3) with $\sigma = 1$ was obtained by [15,16].

Recently, some properties of solutions to the Camassa–Holm equation have been studied by many authors. It is well-known that classical solutions either exist for all time or develop singularities in finite time in the form of breaking waves, that is, the solution remains bounded while its slope becomes unbounded, see the discussion in the paper [17]. Moreover, after wave breaking the solution can be continued as a global weak solution, see [8]. Another point of interest is the existence of peakons, peaked travelling waves similar to the waves of greatest height encountered in classical hydrodynamics, see the discussion in the papers [18,19]. Himonas et al. [20] studied the persistence properties and unique continuation of solutions of the Camassa–Holm equation. They showed that a strong solution of the Camassa–Holm equation, initially decaying exponentially together with its spacial derivative, must be identically equal to zero if it also decays exponentially at a later time, see [21–24] for the similar properties of solutions to other shallow water equation. Just recently, Himonas–Kenig [25] and Himonas et al. [26] considered the non-uniform dependence on initial data for the Camassa–Holm equation on the line and on the circle, respectively. Lv et al. [27] obtained the non-uniform dependence on initial data for μ-b equation. Lv–Wang [28] considered the system (1.3) with $\rho = \gamma - \gamma_{xx}$ and obtained the non-uniform dependence on initial data.

In this paper, we will consider the non-uniform dependence on initial data to system (1.1). We remark that there is significant difference between system (1.1) and (1.3) with $\rho = \gamma - \gamma_{xx}$. It is easy to see that when $\rho = \gamma - \gamma_{xx}$, there are some similar properties between the two equations in system (1.3). Thus the proof of non-uniform dependence on initial data to system (1.3) with $\rho = \gamma - \gamma_{xx}$ is similar to the signal equation, for example, the Camassa–Holm equation. But in system (1.1), ρ and u have different properties, see Theorem 2.1. This needs construct different asymptotic solutions, see Section 3. On the other hand, in order to obtain the non-uniform dependence on initial data to system (1.1), we must modify the earlier method used in [25,28]. In papers [25,28], they estimated the H^1-norm of the difference between the approximate and actual solutions, but it does not work for system (1.1). It is easy to see that the method used in [25,28] is available only when $k_1 = 2$.

This paper is organized as follows. In Section 2, we recall the well-posedness result of Liu–Yin [1] and use it to prove the basic energy estimate from which we derive a lower bound for the lifespan of the solution as well as an estimate of the $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ norm of the solution $(u(t,x), \rho(t,x))$ in terms of $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ norm of the initial data (u_0, ρ_0). In Section 3, we construct approximate solutions, compute the error and estimate the H^μ-norm of this error. In Section 4, we estimate the difference between approximate and actual solutions, where the exact solution is a solution to system (1.1) with initial data given by the approximate solutions evaluated at time zero. The non-uniform dependence on initial data for system (1.1) is established in Section 5 by constructing two sequences of solutions to (1.1) in a bounded subset of the Sobolev space $H^s(\mathbb{R})$, whose distance at the initial time is converging to zero while at any later time it is bounded below by a positive constant.

Notation. In the following, we denote by $*$ the spatial convolution. Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. Let $[A, B] = AB - BA$ denote the commutator of linear operator A and B. Set $\|z\|^2_{H^s \times H^{s-1}} = \|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}$, where $z = (u, \rho)$.

2. Local well-posedness

In this section we first recall the known results of Liu–Yin [1] and give a new estimate of the solution to (1.1).

Let $\Lambda = (1 - \partial_x^2)^{1/2}$. Then the operator Λ^{-2} acting on $L^2(\mathbb{R})$ can be expressed by its associated Green's function $G(x) = \frac{1}{2}e^{-|x|}$ as

$$\Lambda^{-2}f(x) = (G * f)(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}f(y)\mathrm{d}y, \quad f \in L^2(\mathbb{R}).$$

Hence (1.3) is equivalent to the following system

$$\begin{cases} u_t - uu_x = \partial_x\Lambda^{-2}\left(\dfrac{k_1}{2}u^2 + \dfrac{3-k_1}{2}u_x^2 + \dfrac{k_2}{2}\rho^2\right), & t > 0,\ x \in \mathbb{R}, \\ \rho_t - k_3u\rho_x = k_3u_x\rho, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{2.1}$$

with initial data

$$u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}. \tag{2.2}$$

The following result is given by Liu–Yin [1].