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Applications of Morse theory to some nonlinear elliptic equations with resonance at zero

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ABSTRACT

In this paper we study the existence and multiplicity of solutions for some nonlinear elliptic boundary value problems with resonance at zero by applying Morse theory. We do not impose additional global condition on the nonlinearities, except for a subcritical growth condition.

1. Introduction

In this paper we first consider the following quasilinear elliptic equation

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < +\infty$, and $\lambda \in \mathbb{R}$ is a parameter. Assume

(f_0) $f(x,0) = 0$, and $f(x,u) \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfying the growth condition:

$$|f(x,u)| \le c(1 + |u|^{q-1}), \quad \forall\, x \in \Omega,\ u \in \mathbb{R},$$

for some $c > 0$ and $q \in [1, p^*)$, where $p^* = Np/(N-p)$ if $p < N$ and $p^* = +\infty$ if $N \le p$. It is well known that weak solutions of Eq. (P_1) correspond to critical points of the $\mathcal{C}^1$ functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \int_\Omega F(x,u)dx,$$

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where $F(x,u)=\int_0^u f(x,t)dt$, and $W_0^{1,p}(\Omega)$ is the Sobolev space endowed with the norm

$$\|u\|=\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}.$$

Let λ_1 be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, then we know that λ_1 is simple and has an associated eigenfunction $\varphi_1(x)>0$ for $x\in\Omega$ (see [16]). Let $V=\mathrm{span}\{\varphi_1\}$, and we denote by

$$V^\perp=\left\{u\in W_0^{1,p}(\Omega):\ \int_\Omega (\varphi_1(x))^{p-1}u dx=0\right\},$$

then we have

$$W_0^{1,p}(\Omega)=V\oplus V^\perp. \tag{1.1}$$

Moreover, according to [9], we know that there exists $\overline{\lambda}>\lambda_1$ such that

$$\int_\Omega |\nabla u|^p dx\geq \overline{\lambda}\int_\Omega |u|^p dx,\quad \text{for any } u\in V^\perp.$$

In recent years, extensive research on Eq. (P_1) has been done with the non-resonant or resonant conditions. For example, for $\lambda<\lambda_1$, using (f_0) the author in [12] has proved that the superlinear elliptic equation (P_1) has at least three nontrivial solutions, and the authors in [14] have proved that the same problem has at least one nontrivial solution for $\lambda_1<\lambda<\overline{\lambda}$. In [19], with the following condition

(f_1) there exists $\alpha_1>0$ such that

$$0\leq pF(x,u)<(\overline{\lambda}-\lambda_1)|u|^p,\quad \text{for } |u|\leq\alpha_1,\ x\in\Omega,$$

the authors can prove that the functional I_{λ_1} has a local linking at zero with respect to (1.1), and obtain multiple nontrivial solutions for Eq. (P_1) with double resonance at infinity and at zero. For the existence of solutions obtained via variational or topological methods, we also refer to [1–3,7,11,13,29] and references therein.

Note that, in all quoted papers above, the existence of nontrivial solutions for the resonant problem depends on the interplay of nonlinearity's behaviors near infinity and near zero. However, when $p=2$ the authors in [27] obtain at least two nontrivial solution for $\lambda=\lambda_1$ and one nontrivial solution for the higher eigenvalue $\lambda=\lambda_k$ $(k>2)$, by only assuming the following conditions on the $\mathcal{C}^1$ nonlinearity near zero: $f'(x,0)=0$ for $x\in\Omega$ and

$(f_1^\pm)$ there exists $\alpha_2>0$ such that

$$\pm F(x,u)>0,\quad \text{for } 0<|u|\leq\alpha_2,\ x\in\Omega.$$

The first aim of this paper is to extend the results in [27] to the quasilinear elliptic equation (P_1) under weaker conditions. Precisely, our result reads as follows.

Theorem 1.1. *Suppose that (f_0) and (f_1) hold. If $\lambda=\lambda_1$, then Eq. (P_1) has at least one nontrivial solution.*

Remark 1. Note that, except for the growth condition, we also need no conditions on $f(x,u)$ for u large. Moreover, Theorem 1.1 generalizes the results in [27] in the sense that the space $W_0^{1,p}(\Omega)$ with $p\neq 2$ is not a Hilbert space and the functional I_λ may be only of class $\mathcal{C}^1$.

Next, for the ordinary differential equation we can get more nontrivial solutions. More precisely, consider the two point boundary value problem

$$\begin{cases}-\ddot{u}=\Lambda u+g(x,u), & x\in(0,\pi),\\ u(0)=u(\pi)=0,\end{cases} \tag{P_2}$$

where $\Lambda\in\mathbb{R}$ is a parameter. Assume that $g(x,u):[0,\pi]\times\mathbb{R}\to\mathbb{R}$ is a differentiable function satisfying

$(g^\pm)$ $g(x,0)=g'(x,0)=0$, and there exists $\alpha>0$ such that

$$\pm G(x,u)>0,\quad \text{for } 0<|u|\leq\alpha,\ x\in(0,\pi),$$

where $G(x,u)=\int_0^u g(x,s)ds$. Then we have the following existence result.

Theorem 1.2. *Suppose that (g^+) (or (g^-)) holds. If $\Lambda=m^2$ $(m=1,2,\ldots)$, then Eq. (P_2) has at least two nontrivial solutions.*

Remark 2. (1) Note that, Theorem 1.2 extends the results in [27], which can only get one nontrivial solution for the case of $m>2$. Our proof is based on the following fact that the eigenvalues of the linear boundary problem

$$\begin{cases}-\ddot{u}=\Lambda u, & x\in(0,\pi),\\ u(0)=u(\pi)=0,\end{cases} \tag{1.2}$$

are $\Lambda=m^2$ $(m=1,2,\ldots)$, and they are all simple.