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Some properties on the surfaces of vector fields and its application to the Stokes and Navier–Stokes problems with mixed boundary conditions

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ABSTRACT

In this paper we are concerned with the stationary and non-stationary Stokes and Navier–Stokes problems with mixed boundary conditions involving velocity, pressure, rotation, stress and normal derivative of velocity together. Relying on the relations among strain, rotation, normal derivative of velocity and shape of boundary surface, we get variational formulations of the Stokes and Navier–Stokes problems with the mixed boundary conditions. Also, we study existence and uniqueness of solutions to the problems.

1. Introduction

As mathematical models of flows of incompressible viscous Newtonian fluids the Stokes equations

$$-\nu\Delta v+\nabla p=f,\qquad \nabla\cdot v=0\quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t}-\nu\Delta v+\nabla p=f,\qquad \nabla\cdot v=0\quad \text{in } \Omega \tag{1.2}$$

and Navier–Stokes equations

$$-\nu\Delta v+(v\cdot\nabla)v+\nabla p=f,\qquad \nabla\cdot v=0\quad \text{in } \Omega, \tag{1.3}$$

$$\frac{\partial v}{\partial t}-\nu\Delta v+(v\cdot\nabla)v+\nabla p=f,\qquad \nabla\cdot v=0\quad \text{in } \Omega \tag{1.4}$$

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are used. For these systems, different natural and artificial boundary conditions are considered. For example on solid walls, homogeneous Dirichlet condition $v = 0$ is used. On a free surface a Neumann condition $2\nu\varepsilon(v)n - pn = 0$ may be useful (cf. [28] and references therein). Here and in what follows $\varepsilon(v)$ denotes the so-called strain tensor with the components $\varepsilon_{ij}(v) = \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i)$ and n is the outward normal unit vector.

The Navier slip-with-friction boundary conditions $v \cdot n = 0$, $[\varepsilon(v)n + \alpha v] \cdot \tau = 0$, where and in what follows τ is vector tangent to the boundary, are also used for simulations of flows in the presence of rough boundaries (cf. [47,48,60] and references therein). Combination of the normal component of the velocity and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the tangential component of the velocity and the normal component of the friction (condition for in/out-stream surfaces) (cf. [61]) are frequently used. At the outlet of a channel the boundary condition $\nu\frac{\partial v}{\partial n} - pn = \sigma$ (cf. [20–22,52,53] and references therein) or $\nu\varepsilon(v)n - pn = 0$ (cf. [54,53] and references therein) also is used. Rotation boundary condition has been fairly extensively studied over the past several years. (cf. [14–19,25,43]). Of course, on flat portions of the boundary the rotation boundary condition $v_n|_\Gamma = 0$, $\operatorname{rot} v \times n|_{\Gamma_3} = \phi/\nu$ and the Navier slip condition $v_n|_\Gamma = 0$, $\nu\varepsilon_{n\tau}(v)|_\Gamma = \phi$ are equivalent. (cf. Section 6 in [15], Remark 1.1 in [16]). Also, on a part of boundary one deals with the total pressure (Bernoulli's pressure) $\frac{1}{2}|v|^2 + p$ (cf. [23,24]) or static pressure p (cf. [4,59]). It is also known that the total stress $\sigma^{\text{tot}}(v, p) \cdot n$ on the boundary is a natural boundary condition, where $\sigma^{\text{tot}}(v, p) = -\left(p + \frac{1}{2}|v|^2\right) + 2\nu\varepsilon(v)$ (see [38,39]).

In practice we deal with a mixture of some kind of boundary conditions. There are vast literatures for the Stokes and Navier–Stokes problem with mixed boundary conditions and several variational formulations are used for them.

First, we briefly outline main points of variational formulations for the Stokes and Navier–Stokes problems with mixed boundary conditions.

In the case of mixed boundary conditions of Dirichlet condition and $\nu\frac{\partial v}{\partial n} - pn = \sigma$, for variational formulation of the problem the bilinear form

$$a(v, u) = (\nabla v, \nabla u)_{\mathbf{L}_2(\Omega)} \quad \text{for } v, u \in \mathbf{H}^1(\Omega), \tag{1.5}$$

which is reduced from $-(\Delta v, u)_{\mathbf{L}_2(\Omega)}$ by integration by parts, is used (cf. [9,21,54,56,67]).

When one deals with the mixed boundary conditions of velocity and the component of strain $\varepsilon(u)n$, for variational formulation of the problem the bilinear form

$$a(v, u) = 2\Sigma_{i,j}(\varepsilon_{ij}(v), \varepsilon_{ij}(u))_{L_2(\Omega)} \quad \text{for } v, u \in \mathbf{H}^1(\Omega) \tag{1.6}$$

is used and Korn's inequality is on the base, because in the process of change from $-(\Delta v, u)_{\mathbf{L}_2(\Omega)}$ to $2\Sigma_{i,j}(\varepsilon_{ij}(v), \varepsilon_{ij}(u))_{L_2(\Omega)}$ components of strain are reflected on the boundary integral (see (2.16)). In this way different problems are studied (cf. [7,8,10,34,36,38–42,55,61,62,66]). In [36] another equivalent variational formulation, where strain, pressure, velocity and rotation are unknown functions, also is given.

On the other hand, when one deals with the mixed boundary conditions of the Dirichlet condition and the condition for pressure or rotation on a part of boundary, the bilinear form

$$a(v, u) = (\operatorname{rot} v, \operatorname{rot} u)_{\mathbf{L}_2(\Omega)} \quad \text{for } v, u \in \mathbf{H}^1(\Omega) \tag{1.7}$$

is used for variational formulation of the problem. Because, when $-(\Delta v, u)_{\mathbf{L}_2(\Omega)}$ is reduced to (1.7), is reflected rotation in the boundary integral on the part where flow is tangent and on the part where flow is orthogonal to the boundary the boundary integral disappear, and so by integration by parts $(\nabla p, u)$ pressure is reflected in variational formulation (cf. (2.13), (3.6)). In this case equivalence between the norms $\|v\|_{\mathbf{H}^1(\Omega)}$ and $\|\operatorname{rot} v\|_{\mathbf{L}_2(\Omega)}$ under some conditions (see Theorem A.1 in [31], Lemma 2 in [49]) is on the base and in this way many problems are studied (cf. [1,6,12,13,23–25,29,31,32,49,58,59]). In Section 1 of [29], the bilinear form (1.5) instead of (1.7) is used since two bilinear forms (1.5) and (1.7) for polygon or polyhedral domain under some boundary conditions are equal (cf. [33]). When one deals with the boundary condition for pressure or rotation on a part of boundary, there are other variational formulations using three unknown functions v, p and ω, where $\omega = \operatorname{rot} v$, (cf. [5,2–4]) for the 2-dimensional case and v, p and vector potential for the 3-dimensional case (cf. [44]).

No one considered mixed boundary problems for the Stokes and Navier–Stokes equations with stress and pressure conditions together.

In the present paper, we are concerned with the systems (1.1)–(1.4) with mixed boundary conditions involving Dirichlet, pressure, rotation, stress and normal derivative of velocity together.

Mainly relying on the bilinear form (1.6), we reflect all these boundary conditions into variational formulations of problems. Using the variational formulations we prove existence and uniqueness of weak solutions to boundary value problems under conditions for boundary surface. For the initial boundary value problems we prove existence of weak solutions to the problems without assumption for form of boundary surfaces.

This paper consists of 5 sections and an Appendix.

In Section 2, relations among strain, rotation, normal derivative of vector field and shape of boundary surface are considered. Theorem 2.1 gives relations among strain, rotation of vector fields given near a surface and the shape operator of surface, i.e. the matrix in the second fundamental form of surface, (the curvature of boundary for 2-D case), when the vector fields are tangent to the surface. The relation (2.1) of Theorem 2.1, a generalization for 3-D case of Lemma 2.1 in [30]