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Averaging problem for weakly coupled nonexpansive control systems[☆]

Marc Quincampoix[a,*], Hayk Sedrakyan[b]

[a] *Laboratoire de Mathématiques de Bretagne Atlantique, UMR6205, Université de Brest, 6 Avenue Le Gorgeu, 29200 Brest, France*

[b] *CNRS, IMJ-PRG, UMR 7586, Sorbonne Universits, UPMC Univ Paris 06, UniveParis Diderot, Sorbonne Paris Cit, Case 247, 4 place Jussieu, 75252 Paris, France*

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ABSTRACT

This paper investigates the limit behavior of a singularly perturbed control system with two state variables which are weakly coupled. The main novelty of our averaging approach lies in the fact that the limit dynamic may depend the initial condition of the fast system while in the literature this problem has been usually addressed under conditions that ensure that the limit dynamic is independent to this initial condition. Our study is based on a suitable nonexpansivity condition on the fast system which generalized dissipativity or stability properties of the fast dynamics.

1. Introduction

We consider the following singular control problem with a slow and fast motion

$$\begin{cases} \dot{z_\varepsilon}(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)), \quad u_\varepsilon(t) \in U \\ \varepsilon \dot{y_\varepsilon}(t) = g(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0, \end{cases} \tag{1.1}$$

where U is a metric space, $t \in [0, T]$, $f : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$.

In the above $\varepsilon > 0$ is the small singular perturbation parameter, $t \in [0, T]$ is the time variable, $z_\varepsilon(\cdot) : [0, T] \to \mathbb{R}^m$ is the slow motion, $y_\varepsilon(\cdot) : [0, T] \to \mathbb{R}^n$ is the fast motion, $u_\varepsilon(\cdot)$ is the control function taking values in U and $z_\varepsilon(0) = z_0 \in \mathbb{R}^m$, $y_\varepsilon(0) = y_0 \in \mathbb{R}^n$ are the initial values.

We are interested by the behavior of trajectories when the parameter ε tends to zero. There is a wide literature on this question. There are mainly two kind of approaches of the problem. The first one, the so called "reduction method" consists

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* Corresponding author.
E-mail address: Marc.Quincampoix@univ-brest.fr (M. Quincampoix).

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in reducing the singularly perturbed equation to an algebraic–differential equation.

$$\begin{cases} \dot{z}(t) = f(y(t), z(t), u(t)) \\ 0 = g(y(t), z(t), u(t)) \end{cases} \tag{1.2}$$

and to prove that solutions of (1.1) converge to solution of (1.2). This approach has many applications since the pioneering work of Tichonov [19]. Unfortunately this method requires strong stability assumptions for the second equation of (1.1) (cf also [6,12–15,20].

The second approach is the averaging method that we use in the present paper. It consists in finding a limit dynamical system only for the z variable and to prove the convergence. Let us explain this method now (cf [1,2,7,9–11,17,18]). For doing this we recall the main result of [10] (cf also [9]).

For any $z \in \mathbb{R}^m$, we consider the following associated z-system

$$\begin{cases} \dot{y}(t) = g(y(t), z, u(t)) \\ y(0) = z_0, \end{cases} \tag{1.3}$$

(which solution is denoted by $y^z(\cdot, y_0, u)$) and define the following set-valued map

$$F(S, y_0, z) \doteq cl \bigcup_{u \in U} \left\{ \frac{1}{S} \int_0^S f(y^z(s, y_0, u), z, u(s))\, \mathrm{d}s \right\}.$$

Under suitable assumptions, it is possible to prove that $F(S, y_0, z)$ converge (when $S \to \infty$) to some $\bar{F}(y_0, z)$. The main result of [10] shows that if $F(S, y_0, z)$ converge (when $S \to \infty$) uniformly in y_0, z to a $\bar{F}(z)$ which is independent of y_0, then the trajectories of the differential inclusion

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t)) \\ z(0) = z_0, \end{cases} \tag{1.4}$$

are limit of $z_\varepsilon(\cdot)$ solutions of (1.1) and that conversely any solution $z_\varepsilon(\cdot)$ to (1.1) can be approximated by a solution of (1.4).

The main aim of our work consists in investigating a case where the limit equation

$$\begin{cases} \dot{z}(t) \in \bar{F}(z(t), y_0) \\ z(0) = z_0, \end{cases} \tag{1.5}$$

could depend on y_0. This requires a slightly different approach and technics than [9,10]. This is motivated by a recent result on nonexpansive control [16]. For doing this we restrict our consideration to the following weakly coupled case (also studied in [11])

$$\begin{cases} \dot{z_\varepsilon}(t) = f(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t)), \quad u_\varepsilon(t) \in U \\ \varepsilon \dot{y_\varepsilon}(t) = g(y_\varepsilon(t), u_\varepsilon(t)) \\ z_\varepsilon(0) = z_0 \\ y_\varepsilon(0) = y_0. \end{cases} \tag{1.6}$$

We suppose throughout the paper that there exists a compact set $M \times N$ such that for all $\varepsilon > 0$, the set $M \times N$ invariant for the dynamics of (1.6).

Furthermore we assume a nonexpansivity condition on the map g. Our main result says that the limit trajectories $z_\varepsilon(\cdot)$ of (1.6) are solutions to (1.5). But in contrast to results of [10,9], in general the trajectories of (1.5) do not approximate the solution $z_\varepsilon(\cdot)$ of (1.6). We illustrate this phenomenon by discussing an example.

This paper is organized as follows: the main notations, assumptions and results are stated in Section 2. We motivate the averaging method for singularly perturbed control system in Section 3 and prove the main approximation theorem of the slow motion in Section 4. In Section 5 we discuss two examples: The first one illustrate the fact that the limit field $\bar{F}$ may depend on the initial value y_0. The second show a case where some solutions of the limit differential inclusion (1.5) cannot be approximated by the solutions of (1.1).

2. Main results

We are interested by the limit behavior of the slow motion when the perturbation parameter $\varepsilon > 0$ tends to zero for singularly perturbed control system (SPCS in short) on the bounded time interval $[0, T]$. Denote by $\mathcal{U}$ the set of measurable controls from $\mathbb{R}_+$ to a given nonempty metric space U. The notation B stands for the closed unit ball in a metric space.

Let us consider the following system

$$\begin{cases} \dot{y}(t) = g(y(t), u(t)) \\ y(0) = y_0, \end{cases} \tag{2.1}$$

the unique solution of which is denoted by $t \mapsto y(t, y_0, u)$.