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On the existence, uniqueness and regularity of solutions to the phase-field system with a general regular potential and a general class of nonlinear and non-homogeneous boundary conditions

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ABSTRACT

This paper studies a Caginalp phase-field transition system endowed with a general regular potential, as well as a general class, in both unknown functions, of nonlinear and non-homogeneous (depending on time and space variables) boundary conditions. We first prove the existence, uniqueness and regularity of solutions to the Allen–Cahn equation, subject to the nonlinear and non-homogeneous dynamic boundary conditions. The existence, uniqueness and regularity of solutions to the Caginalp system in this new formulation are also proved. This extends previous works concerned with regular potential and nonlinear boundary conditions, allowing the present mathematical model to better approximate the real physical phenomena, especially phase transitions.

1. Introduction

We consider, in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, with a C^2 boundary $\partial\Omega = \Gamma$ and for a finite time $T > 0$, the following nonlinear parabolic system:

$$\begin{cases} C_p \dfrac{\partial}{\partial t} u + \dfrac{\ell}{2} \dfrac{\partial}{\partial t} \varphi = k \Delta u + f_1(t, x) & \text{in } Q \\ \alpha\xi \dfrac{\partial}{\partial t} \varphi = \xi \Delta \varphi + F(\varphi) + s_\xi u + f_2(t, x) & \text{in } Q, \end{cases} \tag{1.1}$$

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with nonlinear and non-homogeneous Cauchy–Neumann boundary conditions on the unknown function u and nonlinear and non-homogeneous dynamic boundary conditions on the unknown function φ, namely:

$$\begin{cases} k\dfrac{\partial}{\partial \nu}u + hu + g_1(t,x,u) = w_1(t,x) & \text{on } \Sigma \\ \xi\dfrac{\partial}{\partial \nu}\varphi + \alpha\xi\dfrac{\partial}{\partial t}\varphi - \Delta_\Gamma\varphi + c_0\varphi + g_2(t,x,\varphi) = w_2(t,x) & \text{on } \Sigma, \end{cases} \tag{1.2}$$

and with the initial conditions

$$u(0,x) = u_0(x), \qquad \varphi(0,x) = \varphi_0(x) \quad \text{in } \Omega, \tag{1.3}$$

where $Q = (0,T] \times \Omega$, $\Sigma = (0,T] \times \partial\Omega$, and:

- $u(t,x)$ represents the *reduced temperature distribution* in Q;
- $\varphi(t,x)$ is the *phase function* (the order parameter), used to distinguish between the states (phases) of a material which occupies the region Ω at every time $t \in [0,T]$;
- $C_p = \rho c$ (ρ is the *density*, c is the *specific heat capacity*), ℓ, k, α, ξ, h are physical parameters representing: the *latent heat*, the *thermal conductivity*, the *relaxation time*, the *measure of the interface thickness*, the *heat transfer coefficient*, respectively, while $s_\xi = \frac{m[S]_E}{2\sigma}T_E$ is a bounded and positive quantity expressed by positive and bounded physical parameters (see [5]);
- c_0 is a positive constant and Δ_Γ is the Laplace–Beltrami operator;
- $f_1 \in L^p(Q), f_2 \in L^q(Q)$ are given functions (which can be interpreted as *distributed controls*), where p and q satisfy (see also [2,15,16,18,17])

$$q \geq p \geq 2; \tag{1.4}$$

- $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a real function having the structure

$$F(\varphi) = f(\varphi) - a_s|\varphi|^{s-1}\varphi, \quad \forall \varphi \in \mathbb{R}, \tag{1.5}$$

  with $a_s > 0$ and $s \geq 3$ satisfying (see also relation (1.14))

$$\frac{n+2}{n+2-2p} > s \quad \text{if } \frac{1}{p} - \frac{2}{n+2} > 0, \tag{1.6}$$

  while $f(\varphi) \in C^1(\mathbb{R})$ fulfills, for constants $b_1, b_2 > 0$, the following properties:

$$|f'(\varphi)| \leq b_1(1 + |\varphi|^{s-2}), \quad \forall \varphi \in \mathbb{R} \tag{1.7}$$

  and

$$(f(\varphi_1) - f(\varphi_2))(\varphi_1 - \varphi_2) \leq b_2(\varphi_1 - \varphi_2)^2, \quad \forall \varphi_1, \varphi_2 \in \mathbb{R}. \tag{1.7'}$$

- Examples of nonlinearities F depending on t, x and φ can be found in [9,16]. Here we take $F(\varphi)$ independent of the space variable because the main difficulty in treating the parabolic nonlinear problem (1.1) lies in the nonlinearity with respect to φ (see [5,7,15,17,19] and references therein);
- $w_1, w_2 \in W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)$ are given functions (which can be interpreted as *boundary controls*);
- $g_i : \Sigma \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are Carathéodory functions, i.e., $g_i(\cdot,\cdot,z) : \Sigma \to \mathbb{R}$ is measurable, $\forall z \in \mathbb{R}$, and $g_i(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, $\forall (t,x) \in \Sigma$, with $g_i(\cdot,\cdot,0) \in L^\infty(\Sigma)$. Moreover, the following hypotheses are assumed to be satisfied ($i = 1, 2$):

  G_1: $\Big(g_i(t,x,z_1) - g_i(t,x,z_2)\Big)(z_1 - z_2) \geq c_1(z_1 - z_2)^2, \ \forall (t,x) \in \Sigma, z_1, z_2 \in \mathbb{R}$, for a constant $c_1 > 0$;

  G_2: there is a function $\bar{G} : \Sigma \times \mathbb{R}^2 \to \mathbb{R}$ verifying the relations

$$\Big(g_i(t,x,z_1) - g_i(t,x,z_2)\Big)^2 \leq \bar{G}(t,x,z_1,z_2)(z_1 - z_2)^2,$$

$$\bar{G}(t,x,z_1,z_2) \leq c_2(1 + |z_1|^{2(r'-1)} + |z_2|^{2(r'-1)}), \quad \forall (t,x) \in \Sigma, \ z_1, z_2 \in \mathbb{R},$$

  for a constant $c_2 > 0$ and $r' \geq 1$ such that (see relation (2.8))

$$\frac{n+2}{n+2-2p} \geq r' \quad \text{if } \frac{1}{p} - \frac{2}{n+2} > 0; \tag{1.8}$$

  G_3: $g_i(t,x,z)z \geq c_3 z^2, \ \forall (t,x) \in \Sigma, \ z \in \mathbb{R}$, with $c_3 > 0$.

- $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, with $k\frac{\partial}{\partial \nu}u_0 + hu_0 + g_1(0,x,u_0) = w_1(0,x)$, and $\varphi_0 \in W_q^{2-\frac{2}{q}}(\Omega)$, with $\xi\frac{\partial}{\partial \nu}\varphi_0 - \Delta_\Gamma\varphi_0 + c_0\varphi_0 + g_2(0,x,\varphi_0) = w_2(0,x)$.

  Let us point out the following remark regarding the nonlinearity $F(\varphi)$ in (1.1), whose form is defined by (1.5).