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Higher-order asymptotic attraction of pullback attractors for a reaction–diffusion equation in non-cylindrical domains

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ABSTRACT

In this paper, we consider the higher-order asymptotic attraction of pullback attractors for a reaction–diffusion equation with domains expanding in time. Firstly, to make the test functions meaningful, a maximum principle is proved for the corresponding variational solution; then, we establish some higher-order integrability results about the difference of variational solutions, and finally we prove the main result that the known (L^2, L^2) pullback $\mathscr{D}_\lambda$-attractor can attract indeed the $\mathscr{D}_\lambda$ class in L^q-norm for any $q \in [2, +\infty)$.

1. Introduction

Let $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ be a family of nonempty bounded open subsets of $\mathbb{R}^N$ such that

$$\mathcal{O}_s \subset \mathcal{O}_t, \quad s < t. \tag{1.1}$$

Define

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\} \quad \text{and} \quad \tilde{Q}_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_T \times \{t\} \quad \text{for any } T > \tau \tag{1.2}$$

and

$$Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\}, \quad \forall\, \tau \in \mathbb{R},$$

$$\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\}, \qquad \Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}, \quad \forall\, \tau < T.$$

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We consider the following initial boundary value problem with homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_\tau, \\ u = 0 \quad \text{on } \Sigma_\tau, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{1.3}$$

where $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ and $f : Q_\tau \to \mathbb{R}$ are given for $\tau \in \mathbb{R}$, and $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the conditions: there exist nonnegative constants $\alpha_1, \alpha_2, \beta, l$ and $p \geqslant 2$, such that

$$-\beta + \alpha_1 |s|^p \leqslant g(s)s \leqslant \beta + \alpha_2 |s|^p, \quad \forall s \in \mathbb{R} \tag{1.4}$$

and

$$g'(s) \geqslant -l, \quad \forall s \in \mathbb{R}. \tag{1.5}$$

Later observe that, there exist nonnegative constants $\tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\beta}$ such that

$$-\tilde{\beta} + \tilde{\alpha_1} |s|^p \leq G(s) \leq \tilde{\beta} + \tilde{\alpha_2} |s|^p, \quad \forall s \in \mathbb{R}, \tag{1.6}$$

where

$$G(s) := \int_0^s g(r) dr.$$

For each $T > \tau$, consider the auxiliary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_{\tau,T}, \\ u = 0 \quad \text{on } \Sigma_{\tau,T}, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{1.7}$$

where $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ for $\tau \in \mathbb{R}$, g satisfies (1.4)–(1.5) and $f \in L^2_{\text{loc}}(\mathbb{R}; L^2(\mathcal{O}_t))$.

It is well known that the long-time behavior of non-autonomous dissipative dynamical systems can be described in terms of the so-called pullback attractor. For the case of cylindrical domains, that is, $\mathcal{O}_t \equiv \mathcal{O}$ for any $t \in \mathbb{R}$, the existence together with the analysis of its further properties about the pullback attractors for non-autonomous reaction–diffusion equations received much more attention in the past two decades, see [1–5] and the references therein.

On the other hand, the problems with domains changing in time aroused many authors' interest as well, for example, see [6–12] and so on. Recently, under the condition of bounded domains increasing with respect to time, that is, the condition (1.1), Kloeden, Marín-Rubio and Real considered in the pioneer work [9], applying the penalty method, the existence and uniqueness of a variational solution for (1.3) as well as the existence of a (L^2, L^2) pullback $\mathscr{D}_\lambda$-attractor for the dynamical system generated by (1.3) has been established.

In this paper, we are interested in the dynamics of the process generated by the variational solutions of (1.3). More precisely, we will prove (under the similar, slightly weaker, assumptions assumed in [9]) that the (L^2, L^2) pullback $\mathscr{D}_\lambda$-attractor obtained in [9] can attract indeed the $\mathscr{D}_\lambda$ class in the topology of L^q for any $q \in [2, +\infty)$; see Theorems 3.2 and 5.3.

In general, for the dissipative reaction–diffusion equations with nonlinear term satisfies conditions like (1.4)–(1.5), the existence of attractors in L^2 topology can be obtained by a quite standard method, e.g., see [13–15]. However, to obtain further attraction in some L^q topology as q large is not easy for the non-autonomous problems, especially for the case that the power p in (1.4) is large and the spatial dimension $N \geqslant 3$. For example, to the best of our knowledge, even in cylindrical domains, the best result about L^q-type pullback attractor is in L^p topology (p is the same power as that in (1.4)) that was given by Łukaszewicz in [5] by applying some new Gronwall lemmas introduced in [4].

Note that the cylindrical domain problem is a special case of (1.3), thus our results (e.g. Theorem 5.3) imply immediately that both the (L^2, L^p) pullback $\mathscr{D}$-attractor obtained in Łukaszewicz [5] and the (L^2, L^2) pullback $\mathscr{D}$-attractor obtained in Anguiano et al. [2] can attract indeed the $\mathscr{D}$ class in $L^{2+\delta}$-norm for any $\delta \in [0, \infty)$. Hence, our results are interesting even for the cylindrical domain problems.

This paper is organized as follows. In Section 2, we recall firstly some notation and results about variational solutions that obtained in [9], and then the notation about pullback attractor. In Section 3, since our assumption on the forcing term f is slightly weaker than that in [9], for rigorous, we give in this section a sketch proof about the existence of a (L^2, L^2) pullback $\mathscr{D}_\lambda$ attractor, see Theorem 3.2. In Section 4, as that in [12], to make the test functions used in Section 5 meaningful, a maximum principle has been established for the variational solutions, see Theorem 4.1. Finally, our main result, Theorem 5.3, will be proved in Section 5.2.