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Direction and stability of bifurcating solutions for a Signorini problem

Jan Eisner[a,*], Milan Kučera[b,c], Lutz Recke[d]

[a] Institute of Mathematics and Biomathematics, Faculty of Science, University of South Bohemia, Branišovská 1760, 370 05 České Budějovice, Czech Republic

[b] Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Prague 1, Czech Republic

[c] Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia in Pilsen, Univerzitní 8, 30614 Plzeň, Czech Republic

[d] Institute of Mathematics of the Humboldt University of Berlin, Unter den Linden 6, 10099 Berlin, Germany

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ABSTRACT

The equation $\Delta u + \lambda u + g(\lambda, u)u = 0$ is considered in a bounded domain in $\mathbb{R}^2$ with a Signorini condition on a straight part of the boundary and with mixed boundary conditions on the rest of the boundary. It is assumed that $g(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$, λ is a bifurcation parameter. A given eigenvalue of the linearized equation with the same boundary conditions is considered. A smooth local bifurcation branch of non-trivial solutions emanating at λ_0 from trivial solutions is studied. We show that to know a direction of the bifurcating branch it is sufficient to determine the sign of a simple expression involving the corresponding eigenfunction u_0. In the case when λ_0 is the first eigenvalue and the branch goes to the right, we show that the bifurcating solutions are asymptotically stable in $W^{1,2}$-norm. The stability of the trivial solution is also studied and an exchange of stability is obtained.

1. Introduction

This paper concerns the questions of a direction of bifurcation branches and of stability of solutions to Signorini boundary value problems of the type

$$\Delta u + \lambda u + g(\lambda, u)u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_D, \qquad \partial_\nu u = 0 \quad \text{on } \Gamma_N, \tag{1.2}$$

$$u \le 0, \qquad \partial_\nu u \le 0, \qquad u\partial_\nu u = 0 \quad \text{on } \Gamma_U. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbb{R}^2$, Γ_D, Γ_N, Γ_U are parts of its boundary, Γ_U being a flat segment (see Section 2) and ∂_ν denotes the outer normal derivative. It will be always assumed that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-smooth function such that

$$g(\lambda, 0) = 0 \quad \text{for all } \lambda \in \mathbb{R}. \tag{1.4}$$

* Corresponding author.
E-mail addresses: jeisner@prf.jcu.cz (J. Eisner), kucera@math.cas.cz (M. Kučera), recke@mathematik.hu-berlin.de (L. Recke).

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We suppose that there are given an eigenvalue $\lambda_0 > 0$ and a corresponding eigenfunction u_0 to the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \tag{1.5}$$

with the nonlinear boundary conditions (1.2), (1.3).

All solutions are understood in the weak sense as solutions of a variational inequality on the cone $K := \{u \in W^{1,2}(\Omega) : u \leq 0 \text{ on } \Gamma_U, u = 0 \text{ on } \Gamma_D\}$. See Section 2, which summarizes basic assumptions and notation used. Main results are given in Sections 3 and 4 (Theorems 3.1, 4.2 and 4.12).

In Section 3 we suppose that there is a smooth branch $\lambda = \hat{\lambda}(s)$, $u = \hat{u}(s)$ (parametrized by $s \in [0, s_0)$ with $s_0 > 0$) of solutions to (1.1), (1.2), (1.3) with $\hat{\lambda}(0) = \lambda_0$, $\hat{u}(0) = 0$ and $\hat{u}'(0) = u_0$, see Assumption (ESB). In [4] we have proved the existence of such smooth branches for a particular case when Ω is a rectangle and under certain "activity conditions" on the eigenfunction u_0. Such smooth branches exist also in more general situations, but it is complicated to formulate and verify sufficient conditions for their existence. However, we do not need such assumptions for our study of bifurcation direction, and therefore we simply assume that a smooth branch exists. Roughly speaking, we show that it is sufficient to verify the inequalities (3.12) or (3.13) in order to know if $\hat{\lambda}'(0) > 0$ or $\hat{\lambda}'(0) < 0$ (Theorem 3.1).

Section 4 concerns the question of stability of solutions to (1.1), (1.2), (1.3) as stationary solutions to the corresponding evolution problem

$$\partial_t u = \Delta u + \lambda u + g(\lambda, u)u \tag{1.6}$$

with the Signorini boundary conditions (1.2), (1.3) (where ∂_t denotes the partial derivative with respect to time t). We consider only the smallest eigenvalue λ_0 of (1.5), (1.2), (1.3). In this case $u_0 < 0$ on Γ_U, therefore λ_0 is simultaneously the smallest eigenvalue of the problem (1.5) with the classical boundary conditions

$$u = 0 \quad \text{on } \Gamma_D, \qquad \partial_\nu u = 0 \quad \text{on } \Gamma_N \cup \Gamma_U. \tag{1.7}$$

Due to Crandall–Rabinowitz bifurcation theorem there exists a smooth local branch of non-trivial solutions to (1.1), (1.7) emanating at λ_0 from trivial solutions. It consists of two half-branches bifurcating in the direction u_0 and $-u_0$. The half-branch bifurcating in the direction u_0 is simultaneously a branch of solutions to the Signorini boundary value problem (1.1), (1.2), (1.3). The well-known principle of exchange of stability (see, e.g. [2, Theorem 1.16], [10, Section II.8] and [12, Section I.7]) yields that if this half-branch goes to the right from λ_0 then it consists of solutions which are stable as stationary solutions to (1.6) with the classical boundary conditions (1.7). In general, stability in $W^{1,2}(\Omega)$ of a stationary solution u_* to the classical problem (1.6), (1.7) does not imply stability in $W^{1,2}(\Omega)$ of u_* as a stationary solution of the unilateral problem to (1.6), (1.2), (1.3). Indeed, if $u(t)$ is a time-dependent solution of the problem (1.6), (1.7) with the initial condition $u(0) \in K$ (in particular $u(0) \in K$ arbitrarily close to u_*) then it can happen, in general, that $u(t) \notin K$ for arbitrarily small times $t > 0$. Therefore the solution of the unilateral problem (1.6), (1.2), (1.3) with the same initial condition (which must satisfy $u(t) \in K$ for all t) can differ from that of the classical problem (1.6), (1.7) (even for initial conditions close to u_*). However, we show by using the stability criterion for variational inequalities [17] that in our particular situation, the $W^{1,2}(\Omega)$-stability of $\hat{u}(s)$ as a stationary solution to (1.6), (1.7) implies $W^{1,2}(\Omega)$-stability of $\hat{u}(s)$ as a stationary solution to (1.6), (1.2), (1.3). In particular, we obtain an exchange of stability for Signorini problem. To our best knowledge, up to now no analogs of the principle of exchange of stability for variational inequalities are known, with the exception of some special cases (for example obstacle problems with finitely many obstacles, see [5,6]).

We do not know any example of a $W^{1,2}(\Omega)$-stable stationary solution u to the evolutionary Signorini problem (1.6), (1.2), (1.3) which does not simultaneously satisfy the classical boundary conditions $u = 0$ on Γ_U or $\partial_\nu u = 0$ on Γ_U. In particular, we do not know any result of the type of exchange of stability in the case when a bifurcation branch of nontrivial solutions to the Signorini problem (1.1), (1.2), (1.3) is not simultaneously a branch of solutions to the corresponding classical boundary value problem (1.1), (1.7). Let us remark that in [5] we have shown an example of a supercritical bifurcation for a variational inequality when the bifurcating non-trivial solutions are stable although they are bifurcating not from the first eigenvalue but from a higher eigenvalue – a certain surprising non-standard case of exchange of stability.

2. Basic assumptions and notation

We will consider a bounded domain Ω in $\mathbb{R}^2$ with a boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_U}$, where $\Gamma_D, \Gamma_N, \Gamma_U$ are pairwise disjoint relatively open subsets of $\partial\Omega$, $\Gamma_D \neq \emptyset$, $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ is finite,

$$\Gamma_U = \{(x, 0) : x \in (\gamma_1, \gamma_2)\}$$

with some $\gamma_1 < \gamma_2$. We will assume that

$$\text{there is } \mu_0 > 0 \quad \text{such that } \Gamma_{N,\mu_0} := \{(x, 0) : x \in (\gamma_1 - \mu_0, \gamma_1) \cup (\gamma_2, \gamma_2 + \mu_0)\} \subset \Gamma_N. \tag{2.1}$$

In particular, Γ_U and its μ_0-neighbourhood in $\partial\Omega$ are supposed to be flat. We introduce a real Hilbert space H with the scalar product $\langle \cdot, \cdot \rangle$, defined by

$$H := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_D\}, \qquad \langle u, \varphi \rangle := \int_\Omega \nabla u \cdot \nabla \varphi \, dx\, dy \quad \text{for } u, \varphi \in H,$$

and with the corresponding norm $\|\cdot\|$ which is equivalent on our space H to the usual Sobolev norm.