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Global integrability for weak solutions to some anisotropic elliptic equations

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ABSTRACT

We consider the boundary value problem

$$\begin{cases} \sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0, & x \in \Omega; \\ u(x) = u_*(x), & x \in \partial\Omega. \end{cases}$$

We show that higher integrability of the boundary datum u_* forces solutions u to have higher integrability as well. Assumptions on $a_i(x, z)$ are suggested by Euler equation of the anisotropic functional

$$\int_\Omega \sum_{i=1}^{n} \left(2|D_i u|^2 + |D_i u| \sin(|D_i u|)\right)^{\frac{p_i}{2}}.$$

1. Introduction

We consider integral functionals

$$\mathcal{I}(u) = \int_\Omega f(x, Du(x))dx \tag{1.1}$$

where $u : \Omega \to \mathbb{R}$, Ω is a bounded open subset of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$; about $f(x, z)$ we assume that $x \to f(x, z)$ is measurable and $z \to f(x, z)$ is continuous; u is taken from Sobolev space $W^{1,1}(\Omega)$. We are interested in functions u solving the Euler equation

$$\sum_{i=1}^{n} D_i \left(\frac{\partial f}{\partial z_i}(x, Du(x))\right) = 0 \tag{1.2}$$

in weak form, or more generally

$$\sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0, \tag{1.3}$$

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where $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ with $x \to a_i(x, z)$ measurable and $z \to a_i(x, z)$ continuous. In past years great attention has been paid to anisotropic functionals whose model is

$$\int_\Omega (|D_1 u|^{p_1} + |D_2 u|^{p_2} + \cdots + |D_n u|^{p_n}) dx \tag{1.4}$$

where the derivative $D_i u = \frac{\partial u}{\partial x_i}$ has the exponent p_i that might be different from the exponent p_j of the derivative $D_j u = \frac{\partial u}{\partial x_j}$, when $j \neq i$. Such a model suggests to consider energies $f(x, z)$ where

$$|f(x, z)| \le c \left(1 + \sum_{i=1}^n |z_i|^{p_i}\right) \tag{1.5}$$

or Eq. (1.3) with coefficients $a_i(x, z)$ satisfying

$$|a_i(x, z)| \le c(1 + |z_i|)^{p_i - 1}. \tag{1.6}$$

This anisotropic framework looks useful when dealing with some reinforced materials, see [15]; about theoretical viewpoint see [10], example 1.7.1, page 169. In the present paper we are interested in the integrability of solutions u to (1.3): does high integrability of boundary datum u_* improve the integrability of the solution u? A positive answer has been given in [8,4,2] when the operator is monotone:

$$\nu \sum_{i=1}^n |z - \tilde{z}|^{p_i} \le \sum_{i=1}^n (a_i(x, z) - a_i(x, \tilde{z}))(z_i - \tilde{z}_i) \tag{1.7}$$

for some positive constant ν. Please, note that monotonicity forces f to be convex, when $a_i(z) = \frac{\partial f}{\partial z_i}(z)$. Recently, [9] shows that convexity of f is not necessary; only coercivity of f is required:

$$\nu_* \sum_{i=1}^n |z|^{p_i} \le f(x, z), \tag{1.8}$$

for some positive constant ν_*. The result contained in [9] is valid for minimizers of (1.1). When f is no longer convex, stationary maps u need not to minimize $\mathcal{I}$, so we cannot use such a result. In the present paper we deal with stationary maps u and we show higher integrability, provided coercivity for $\frac{\partial f}{\partial z}$ is assumed:

$$\tilde{\nu} \sum_{i=1}^n |z|^{p_i} \le \sum_{i=1}^n \frac{\partial f}{\partial z_i}(x, z) z_i, \tag{1.9}$$

for some positive constant $\tilde{\nu}$. More generally, higher integrability holds true for weak solutions u to (1.3) under coercivity for a:

$$\nu \sum_{i=1}^n |z|^{p_i} \le \sum_{i=1}^n a_i(x, z) z_i, \tag{1.10}$$

for some positive constant ν. In order to state our theorem, let us assume that $p_1, \dots, p_n \in (1, +\infty)$ with $\overline{p} < n$, where $\overline{p}$ is the harmonic mean, that is

$$\frac{1}{\overline{p}} \sum_{i=1}^n \frac{1}{p_i}; \tag{1.11}$$

condition $\overline{p} < n$ allows us to consider the Sobolev exponent $\overline{p}^* = \frac{n\overline{p}}{n - \overline{p}}$. As far as the boundary datum u_* is concerned, we assume that

$$u_* \in W^{1,1}(\Omega) \quad \text{with } D_i u_* \in L^{q_i}(\Omega), \quad q_i \in (p_i, +\infty) \tag{1.12}$$

for every $i = 1, \dots, n$. Let us introduce the Sobolev space

$$W_0^{1,(p_i)}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \ \forall i = 1, \dots, n \right\}. \tag{1.13}$$

In this paper we will prove the following