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Existence and uniqueness of global solutions to fully nonlinear first order elliptic systems

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ABSTRACT

Let $F : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ be a Carathéodory map. In this paper we consider the problem of existence and uniqueness of weakly differentiable global strong a.e. solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the fully nonlinear PDE system

$$F(\cdot, Du) = f, \quad \text{a.e. on } \mathbb{R}^n, \tag{1}$$

when $f \in L^2(\mathbb{R}^n)^N$. By introducing an appropriate notion of ellipticity, we prove the existence of solution to (1) in a tailored Sobolev "energy" space (known also as the J.L. Lions space) and a uniqueness a priori estimate. The proof is based on the solvability of the linearised problem by Fourier transform methods and a "perturbation device" which allows the use of Campanato's notion of near operators, an idea developed for the 2nd order case.

1. Introduction

Let $n, N \geq 2$ and let also

$$F : \mathbb{R}^n \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^N,$$

be a Carathéodory map, namely

$$\begin{cases} x \mapsto F(x, Q) \text{ is measurable, for every } Q \in \mathbb{R}^{N\times n}, \\ Q \mapsto F(x, Q) \text{ is continuous, for almost every } x \in \mathbb{R}^n. \end{cases}$$

In this paper we consider the problem of existence and uniqueness of global strong a.e. solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the fully nonlinear PDE system

$$F(\cdot, Du) = f, \quad \text{a.e. on } \mathbb{R}^n. \tag{1.1}$$

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To the best of our knowledge, the above problem has not been considered before in this generality. We will assume that our right hand side f is an L^2 vector function, i.e. $f \in L^2(\mathbb{R}^n)^N$. By introducing an appropriate ellipticity assumption on F, we will prove unique solvability of (1.1) for a weakly differentiable map u, together with a strong a priori estimate. In the above, $Du(x) \in \mathbb{R}^{N\times n}$ denotes the gradient matrix of $u = u_\alpha e^\alpha$, namely $Du = (D_i u_a) e^\alpha \otimes e^i$ and $D_i = \partial/\partial x_i$. Here and in the sequel we employ the summation convention when $i, j, k, \ldots$ run in $\{1, \ldots, n\}$ and $\alpha, \beta, \gamma, \ldots$ run in $\{1, \ldots, N\}$. Evidently, $\{e^i\}$, $\{e^\alpha\}$ and $\{e^\alpha \otimes e^i\}$ denote the standard bases of $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{N\times n}$ respectively.

The simplest case of (1.1) is when F is independent of x and linear in P, that is when

$$F(x, P) = \mathrm{A}_{\alpha\beta j} P_{\beta j} e^\alpha,$$

for a linear operator $\mathrm{A} : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}^N$. Then (1.1) becomes

$$\mathrm{A}_{\alpha\beta j} D_j u_\beta = f_\alpha,$$

which we will write compactly as

$$\mathrm{A} : Du = f. \tag{1.2}$$

The appropriate notion of ellipticity in this case is that the nullspace of the operator A contains no (non-trivial) rank-one lines. This means

$$|\mathrm{A} : \eta \otimes a| > 0, \quad \eta \neq 0,\ a \neq 0. \tag{1.3}$$

The prototypical example is the operator $\mathrm{A} : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^2$ given by

$$\mathrm{A} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

which corresponds to the Cauchy–Riemann differential operator. Linear elliptic first order systems with constant coefficients have been extensively studied in several contexts, since they play an important role in Complex and Harmonic Analysis (see e.g. Buchanan–Gilbert [2], Begehr–Wen [1]), Compensated Compactness and Differential Inclusions (Di Perna [14], Müller [19]), regularity theory of PDE (see Chapter 7 in [18] for Morrey's exposition of the Agmon–Douglis–Nirenberg theory) and Geometric Analysis with differential forms (Csató–Dacorogna–Kneuss [12]).

The fully nonlinear case of (1.1) is much less studied. When F is *coercive instead of elliptic*, the problem is better understood. By using the analytic Baire category method of the Dacorogna–Marcellini [13] which is the "geometric counterpart" of Gromov's Convex Integration, one can prove that, under certain structural and compatibility assumptions, the Dirichlet problem

$$\begin{cases} F(\cdot, Du) = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has *infinitely many* strong a.e. solutions in the Lipschitz space, for $\Omega \subseteq \mathbb{R}^n$ and g Lipschitz. However, ellipticity and coercivity of F are, roughly speaking, mutually exclusive. In particular, the Dirichlet problem (1.4) is not well posed when F is either linear or elliptic. For example, the equation $u' - 1 = 0$ has no Lipschitz solution on $(0, 1)$ for which $u(0) = u(1) = 0$.

Herein we focus on the general system (1.1) and we consider the problem of finding an ellipticity condition which guarantees the existence as well as *uniqueness* of a strong a.e. weakly differentiable solution. In order to avoid the compatibility difficulties which arise in the case of the Dirichlet problem on bounded domains, we will consider the case of global solutions on the whole space. We will also restrict attention to the case of f in $L^2(\mathbb{R}^n)^N$ and $n \geq 3$. This is mostly for technical simplicity and since the case of $n = 2$ has been studied much more extensively. We will prove unique solvability of (1.1) in the "energy" Sobolev space[1]

$$W^{1;2^*,2}(\mathbb{R}^n)^N := \left\{ u \in L^{2^*}(\mathbb{R}^n)^N \mid Du \in L^2(\mathbb{R}^n)^{Nn} \right\} \tag{1.5}$$

where $n \geq 3$ and 2^* is the conjugate Sobolev exponent:

$$2^* = \frac{2n}{n-2}.$$

The technique we follow for (1.1) is based on the solvability of the linear constant coefficient equation (1.2) via the Fourier transform. In Section 2 we prove the existence and uniqueness of a strong global solution $u \in W^{1;2^*,2}(\mathbb{R}^n)^N$ to (1.2), for which we also have an *explicit* integral representation formula for the solution (Theorem 2). The essential idea in order to go from the linear to the fully nonlinear case is a perturbation device inspired from the work of Campanato [4–9] on the second order case of

$$F(\cdot, D^2u) = f. \tag{1.6}$$

[1] We would like to thank the referee of this paper who pointed out to us that (1.5) is called in the literature the "J.L. Lions space".