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Lengths, areas and Lipschitz-type spaces of planar harmonic mappings

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ABSTRACT

In this paper, we give bounds for length and area distortion for harmonic K-quasiconformal mappings, and investigate certain Lipschitz-type spaces on harmonic mappings.

1. Introduction and main results

Let D be a simply connected subdomain of the complex plane $\mathbb{C}$. A complex-valued function f defined in D is called a *harmonic mapping* in D if and only its real and the imaginary parts of f are real harmonic in D. It is known that every harmonic mapping f defined in D admits a decomposition $f = h+\overline{g}$, where h and g are analytic in D. Since the Jacobian J_f of f is given by

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 := |h'|^2 - |g'|^2,$$

f is locally univalent and sense-preserving in D if and only if $|g'(z)| < |h'(z)|$ in D; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in D (see [16]). Let $\mathcal{H}(D)$ denote the class of all sense-preserving harmonic mappings in D. We refer to [7,9] for basic results in the theory of planar harmonic mappings.

For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$. In particular, we use $\mathbb{D}_r$ to denote the disk $\mathbb{D}(0, r)$ and $\mathbb{D}$, the open unit disk $\mathbb{D}_1$. For a harmonic mapping f defined on $\mathbb{D}$, we use the following standard notations:

$$\Lambda_f(z) = \max_{0\leq\theta\leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$$

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and

$$\lambda_f(z) = \min_{0\leq\theta\leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = \big| |f_z(z)| - |f_{\bar{z}}(z)| \big|.$$

We recall that a function $f \in \mathcal{H}(\mathbb{D})$ is said to be K-*quasiregular*, $K \in [1, \infty)$, if for $z \in \mathbb{D}$, $\Lambda_f(z) \leq K\lambda_f(z)$. In addition, if f is univalent in $\mathbb{D}$, then f is called a K-*quasiconformal* harmonic mapping in $\mathbb{D}$.

Let Ω be a domain of $\mathbb{C}$, with non-empty boundary. Let $d_\Omega(z)$ be the Euclidean distance from z to the boundary $\partial\Omega$ of Ω. In particular, we always use $d(z)$ to denote the Euclidean distance from z to the boundary of $\mathbb{D}$. The normalized area of a set $G \subset \mathbb{C}$ is denoted by $A(G)$. It means that $A(G) = \text{area}\,(G)/\pi$, where area (G) is the area of G. The area problem of analytic functions has attracted much attention (see [1,22–24]). We investigate the area problem of harmonic mappings and obtain the following result.

Theorem 1. *Let Ω_1 and Ω_2 be two proper and simply connected subdomains of $\mathbb{C}$ containing the point of origin. Then for a sense-preserving and K-quasiconformal harmonic mapping f defined in Ω_1 with $f(0) = 0$,*

$$KA\big(f(\Omega_1) \cap \Omega_2\big) + A(f^{-1}(\Omega_2)) \geq \min\{d^2_{\Omega_1}(0), d^2_{\Omega_2}(0)\}. \tag{1}$$

Moreover, if $K = 1$, then the estimate of (1) is sharp.

We remark that Theorem 1 is a generalization of [22, Theorem].

A planar harmonic mapping f defined on $\mathbb{D}$ is called a *harmonic Bloch mapping* if

$$\beta_f = \sup_{z,w\in\mathbb{D},\, z\neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < \infty.$$

Here β_f is called the *Lipschitz number* of f, and

$$\rho(z, w) = \frac{1}{2}\log\left(\frac{1 + |(z - w)/(1 - \bar{z}w)|}{1 - |(z - w)/(1 - \bar{z}w)|}\right) = \text{arctanh}\left|\frac{z - w}{1 - \bar{z}w}\right|$$

denotes the hyperbolic distance between z and w in $\mathbb{D}$. It is known that

$$\beta_f = \sup_{z\in\mathbb{D}}\big\{(1 - |z|^2)\Lambda_f(z)\big\}.$$

Clearly, a harmonic Bloch mapping f is uniformly continuous as a map between metric spaces,

$$f : (\mathbb{D}, \rho) \to (\mathbb{C}, |\cdot|),$$

and for all $z, w \in \mathbb{D}$ we have the Lipschitz inequality

$$|f(z) - f(w)| \leq \beta_f\, \rho(z, w)\,.$$

A well-known fact is that the set of all harmonic Bloch mappings, denoted by the symbol $\mathcal{HB}$, forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\|_{\mathcal{HB}} = |f(0)| + \sup_{z\in\mathbb{D}}\{(1 - |z|^2)\Lambda_f(z)\}.$$

Specially, we use $\mathcal{B}$ to denote the set of all analytic functions defined in $\mathbb{D}$ which forms a complex Banach space with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}\{(1 - |z|^2)|f'(z)|\}.$$

The reader is referred to [8, Theorem 2] (see also [5,6]) for a detailed discussion.

For $r \in [0, 1)$, the length of the curve $C(r) = \big\{f(re^{i\theta}) : \theta \in [0, 2\pi]\big\}$, counting multiplicity, is defined by

$$\ell_f(r) = \int_0^{2\pi} |df(re^{i\theta})| = r\int_0^{2\pi} \big|f_z(re^{i\theta}) - e^{-2i\theta} f_{\bar{z}}(re^{i\theta})\big|\, d\theta,$$

where f is a harmonic mapping defined in $\mathbb{D}$. In particular, it is convenient to set

$$\ell_f(1) = \sup_{0<r<1} \ell_f(r).$$

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$ be a sense-preserving K-quasiconformal harmonic mapping. If $\ell_f(1) < \infty$, then for $n \geq 1$,*

$$|a_n| + |b_n| \leq \frac{K\ell_f(1)}{2n\pi} \tag{2}$$

and

$$\Lambda_f(z) \leq \frac{\ell_f(1)\sqrt{K}}{2\pi(1 - |z|)}. \tag{3}$$