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Sharp existence theorems for multiple vortices induced from magnetic impurities

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ABSTRACT

In a recent work Tong and Wong derived self-dual equations governing multiple vortices in a product Abelian Higgs model which may be regarded as a generalized Ginzburg–Landau theory. The purpose of this paper is to establish some sharp existence and uniqueness theorems for these multiple vortex solutions in a doubly periodic domain and on the full plane using the methods of calculus of variations.

1. Introduction

Let ϕ be a complex-valued scalar field defined over $\mathbb{R}^2$. The gauge-covariant derivative is then given by $D_j\phi = \partial_j\phi - \mathrm{i}A_j\phi$, $j = 1, 2$, where A_j is a real-valued vector gauge field which induces a magnetic or vorticity field F_{jk} through $F_{jk} = \partial_j A_k - \partial_k A_j$, $j, k = 1, 2$. The associated total energy is

$$E(\phi, A) = \int_{\mathbb{R}^2} \left(\frac{1}{2}F_{12}^2 + \frac{1}{2}|D_1\phi|^2 + \frac{1}{2}|D_2\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2 \right) \mathrm{d}x, \tag{1.1}$$

where $\lambda > 0$. The Euler–Lagrange equations, or the static Abelian Higgs equations, also known as the Ginzburg–Landau equations, associated with the energy (1.1) are

$$D_k D_k \phi = \frac{\lambda}{2}(|\phi|^2 - 1)\phi, \tag{1.2}$$

$$\partial_k F_{jk} = \frac{\mathrm{i}}{2}(\phi\overline{D_j\phi} - \overline{\phi}D_j\phi), \tag{1.3}$$

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which are still impossible to fully understand. In the special situations $\lambda = 1$ that borders type-I and type-II superconductivity, these equations enjoy a dramatic reduction into the so-called self-dual system of equations of the form

$$D_1\phi + \mathrm{i}D_2\phi = 0, \tag{1.4}$$

$$F_{12} = \frac{1}{2}(1 - |\phi|^2), \tag{1.5}$$

which are also known as the BPS equations, named after Bogomol'nyi [8] and Prasad–Sommerfield [37]. Existence and uniqueness of solutions of these equations have been established by Taubes [26,44,45] on the full plane and by Wang and Yang [49] on a doubly periodic domain to resemble Abrikosov's vortices [1]. Since these works, a lot of results have been obtained and developed in numerous areas including the Chern–Simons models [9–14,16–18,22,25,30,31,34,35,40–43], electroweak theory [2–5,38,39,51], and cosmic strings [19,27,28,48], which have greatly enriched our understanding of field-theoretical vortices.

Recently there have been a lot of activities in the study and development of field theoretical models governed by the supersymmetric defect dynamics. Supersymmetric gauge theories have been used as tractable, toy models to explore strongly coupled phenomena in high energy physics [21,46,47]. It is shown that one can add charged defects into field theories in $d = 2+1$ dimensions, preserving some amount of supersymmetry [23]. This provides the prospect of using supersymmetric methods to study strongly coupled phenomena in the presence of doped impurities or lattices [7]. As in the study of the classical Abelian Higgs model described above, with some special choices of the Higgs potentials in $(2 + 1)$ dimensional gauge models, one can obtain interesting limiting BPS structures [8,37]. The appearance of such structures for certain special Higgs potentials may be ascribed either to extended supersymmetry [20,50] or to suitable dimensional reductions of some 4D self-dual Yang–Mills systems [45]. The solutions of these systems of equations are of the characteristics of topological defects resembling magnetic vortices in a type-II superconductor and have long played an important role in particle physics and condensed matter physics, and arise as a consequence of spontaneous symmetry breaking [53]. In the context of Abelian theories in $d = 2 + 1$ dimensions, the addition of electric and magnetic impurity has been explored recently in [23,24]. In [46], Tong and Wong study the dynamics of the monopoles in the presence of the spin impurities. A very similar problem is solved in [47], giving rise to an onset of the Abelian vortices in the presence of electric impurities. In [47], Tong and Wong give an analysis of more general spatially dependent impurities and their effects on vortices. They argued that a moduli space of solitons survives the addition of both electric and magnetic impurities. In the case of electric impurities, the metric remains unchanged but the dynamics is accompanied by a connection term, acting as an effective magnetic field over the moduli space. In contrast, magnetic impurities distort the metric on the moduli space. It is shown that magnetic impurities can be viewed as vortices associated to a separate, frozen, gauge group. It is also seen [53] that multiple distributed cosmic strings arise in the product Abelian gauge field theory of [47]. In particular, in [53], asymptotic behavior of the string solutions has been precisely described to allow the derivation of a necessary and sufficient condition for the gravitational metric to be geodesically complete and an explicit calculation of the deficit angle proportional to the string tension, both stated in terms of string numbers, energy levels of broken symmetries, and the universal gravitational constant.

In the present paper, we aim at developing a complete existence theory for the multiple BPS vortices in the product Abelian gauge theory model of Tong and Wong [47]. Extending the methods of [15,29,30,32,33,52], we shall establish a series of existence and uniqueness theorems for these multiple vortex solutions under sharp conditions.

The rest of the paper is organized as follows. In Section 2 we present the equations governing the multiple vortices in the product Abelian gauge field theory model of Tong and Wong [47]. In Section 3 we prove the existence and uniqueness of a multiple vortex solution over a doubly periodic domain by a constrained variational method under an explicitly stated necessary and sufficient condition. In Section 4 we prove the existence and uniqueness of a multiple vortex solution realizing an arbitrarily prescribed vortex distribution over $\mathbb{R}^2$.

2. Vortex equations with magnetic impurities

Following Tong and Wong [47], we consider some $U(1)$ gauge theories in $d = 2 + 1$ dimensions. In the theory unspoilt by the presence of dirt, the action is given by

$$S = -\int \mathrm{d}^3x\left[\frac{1}{4e^2}F_{jk}F^{jk} + \frac{1}{2e^2}(\partial_j\phi)^2 + |D_jq|^2 + \frac{e^2}{2}(|q|^2 - \xi)^2 + \phi^2|q|^2\right], \tag{2.1}$$

where the electric field is $E_i = F_{i0}$, the magnetic field is $B = F_{12}$, a scalar q carry charge $+1$, ϕ is a neutral scalar, and $\xi > 0$ is a parameter. The simplest vortex solutions do not involve the neutral scalar ϕ. Setting $E_i = 0$, we can derive first order vortex equations by the BPS trick [8,37]

$$B = e^2(|q|^2 - \xi) \quad \text{and} \quad (D_1 + \mathrm{i}D_2)q \equiv D_zq = 0, \tag{2.2}$$

with $z = x^1 + \mathrm{i}x^2$.

Then we consider the effect of magnetic impurities on vortices. The impurities are comprised of a fixed, static source term $\sigma(x)$ for the magnetic field,

$$S_{\text{impurity}} = -\int \mathrm{d}^3x\sigma(x)B. \tag{2.3}$$