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Energy functionals of Kirchhoff-type problems having multiple global minima

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ABSTRACT

In this paper, using the theory developed in Ricceri (2012), we obtain some results of a totally new type about a class of non-local problems. Here is a sample:

Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain, with $n \geq 4$, let $a, b, \nu \in \mathbf{R}$, with $a \geq 0$ and $b > 0$, and let $p \in]0, \frac{n+2}{n-2}[$.

Then, for each $\lambda > 0$ large enough and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains $H_0^1(\Omega)$, there exists $v^* \in C$ such that the problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u(x)|^2 dx\right) \Delta u = \nu|u|^{p-1}u + \lambda(u - v^*(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the corresponding energy functional.

Introduction

Here and in what follows, $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary and $a, b \in \mathbf{R}$, with $a \geq 0$ and $b > 0$.

Consider the non-local problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u(x)|^2 dx\right) \Delta u = h(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$h : \Omega \times \mathbf{R} \to \mathbf{R}$ being a Carathéodory function.

In the past years, many papers have been produced on the existence and multiplicity of weak solutions for this problem. Usual reference papers are [1,2,4–6,10].

In the present note, we are interested in the multiplicity of solutions of the above problem under the following new aspect which has not been considered before: among the solutions of the problem, at least two are global minima of the energy functional.

But, more generally, the global structure itself of the conclusions we reach is novel at all, so that no proper comparison of our results with the previous ones in the field can be made.

For instance, the following proposition summarizes very well the novelties of our main result (Theorem 1) of which it is the simplest particular case:

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Proposition 1. *Let $n \geq 4$, let $\nu \in \mathbf{R}$ and let $p \in \left]0, \frac{n+2}{n-2}\right[$.*

Then, for each $\lambda > 0$ large enough and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains $H_0^1(\Omega)$, there exists $v^* \in C$ such that the problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = \nu|u|^{p-1}u + \lambda(u - v^*(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional

$$u \to \frac{a}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u(x)|^2 dx\right)^2 - \frac{\nu}{p+1}\int_\Omega |u(x)|^{p+1} dx - \frac{\lambda}{2}\int_\Omega |u(x) - v^*(x)|^2 dx.$$

What allows us to obtain results of this totally new type is the use of theory we have recently developed in [8].

Results

On the Sobolev space $H_0^1(\Omega)$, we consider the scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u(x)\nabla v(x) dx$$

and the induced norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}.$$

We denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|f(x,\xi)|}{1 + |\xi|^p} < +\infty \tag{1}$$

for some $p \in \left]0, \frac{n+2}{n-2}\right[$.

Moreover, we denote by $\tilde{\mathcal{A}}$ the class of all Carathéodory functions $g : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|g(x,\xi)|}{1 + |\xi|^q} < +\infty \tag{2}$$

for some $q \in \left]0, \frac{2}{n-2}\right[$.

Furthermore, we denote by $\hat{\mathcal{A}}$ the class of all functions $h : \Omega \times \mathbf{R} \to \mathbf{R}$ of the type

$$h(x, \xi) = f(x, \xi) + \alpha(x)g(x, \xi)$$

with $f \in \mathcal{A}$, $g \in \tilde{\mathcal{A}}$ and $\alpha \in L^2(\Omega)$. For each $h \in \hat{\mathcal{A}}$, we define the functional $I_h : H_0^1(\Omega) \to \mathbf{R}$, by putting

$$I_h(u) = \int_\Omega H(x, u(x)) dx$$

for all $u \in H_0^1(\Omega)$, where

$$H(x, \xi) = \int_0^\xi h(x, t) dt$$

for all $(x, \xi) \in \Omega \times \mathbf{R}$.

By classical results (involving the Sobolev embedding theorem), the functional I_h turns out to be sequentially weakly continuous, of class C^1, with compact derivative given by

$$I_h'(u)(w) = \int_\Omega h(x, u(x)) w(x) dx$$

for all $u, w \in H_0^1(\Omega)$.

Now, let us recall that, given $h \in \hat{\mathcal{A}}$, a weak solution of the problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = h(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$