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Well-posedness and dynamics of stochastic fractional model for nonlinear optical fiber materials[☆]

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ABSTRACT

The current paper is devoted to the well-posedness and dynamics of the stochastic coupled fractional Ginzburg–Landau equation, which describes a class of nonlinear optical fiber materials with active and passive coupled cores. By the commutation estimates and Fourier–Galerkin approximation, the global existence of weak solutions and the uniqueness criterion are established. Moreover, the existence of a global attractor is shown. Finally, we consider the long-time behavior of the stochastic coupled fractional Ginzburg–Landau equation (SCFGL) with multiplicative noise, and prove the existence of a random attractor for the random dynamical system generated by the SCFGL equation.

1. Fractional model for nonlinear optical fiber materials

Ginzburg–Landau (GL) equations are usually applied to describe a class of optical fiber materials. There has been extensive study of the GL equations (see [1–6] and reference therein). The exact homoclinic wave and soliton solution of the GL equations have been studied in [2]. Guo et al. in [5] proved the existence of a global attractor for the GL equation. The coupled Ginzburg–Landau (CGL) equations have attracted considerable attention in modeling a class of nonlinear optical fiber materials with active and passive coupled cores. There are also many papers concerning the CGL equations (see [7–9] and reference therein). The existence of the stable solutions and exponential attractors for the CGL systems has been proved in [7,9] respectively.

The fractional Laplacian operator is exactly the infinitesimal generators of Lévy stable diffusion processes, and there are many fractional models that arise in plasma, flames propagation and chemical reactions in liquids, geophysical fluid dynamics and financial market et al. There have been extensive study and application of fractional differential equations including the fractional Schrödinger equation [10], fractional Landau–Lifschitz equation [11], fractional Landau–Lifschitz–Maxwell equation [12] and fractional Ginzburg–Landau (FGL) equation [13]. Pu and Guo in [13] proved the well-posedness and dynamics for the FGL equation. However, there are some limitations to this model to describe some models with some perturbations which will lead to a very large complex system. In mathematical physics, the models can be described by stochastic partial differential equations. Based on [13], the dynamics for the stochastic FGL equation with multiplicative noise has been studied in [14].

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Motivated by [13,15], in the present paper, we consider the following coupled fractional Ginzburg–Landau (CFGL) equation:

$$\begin{cases} u_t = \gamma_1 u - (\gamma_2 + i\gamma_3)(-\Delta)^\alpha u + (i\sigma_1 - \sigma_2)|u|^2 u + iv, & x \in \mathbb{R},\ t > 0, \\ v_t = (-\mu_1 + i\mu_2)v - (\mu_3 + i\mu_4)(-\Delta)^\beta v + iu, & x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.1}$$

with the initial conditions and the periodic boundary conditions:

$$\begin{cases} u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}, \\ u(x+2\pi, t) = u(x,t), \quad v(x+2\pi,t) = v(x,t), & t > 0, x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha, \beta \in (0,1)$, u and v denote the amplitude of the electromagnetic wave in a dual-core system, t denotes the time, x is the horizontal axis of the plane wave, $\gamma_2 > 0$, $\mu_2 > 0$, $\mu_3 > 0$ and $\mu_4 > 0$ are dissipation coefficients and $\mu_1 > 1$, γ_1, γ_3, σ_1, σ_2 are real numbers. The fractional Laplacian $(-\Delta)^\alpha$ can be regarded as a pseudo differential operator with $|\xi|^{2\alpha}$ and can be realized through the Fourier transform [16]:

$$\widehat{(-\Delta)^\alpha} u(\xi) = |\xi|^{2\alpha}\widehat{u}(\xi), \tag{1.3}$$

where $\widehat{u}$ is the Fourier transform of u. In what follows, we write Λ for $(-\Delta)^{\frac{1}{2}}$.

However, there is a natural question: How about the dynamics for the stochastic CFGL equations? Motivated by [14], we also consider the following stochastic CFGL equation with multiplicative noise:

$$\begin{cases} du = [\gamma_1 u - (\gamma_2 + i\gamma_3)(-\Delta)^\alpha u + (i\sigma_1 - \sigma_2)|u|^2 u + iv]dt + \beta_1 u dW_1(t), & x \in \mathbb{R},\ t > 0, \\ dv = [(-\mu_1 + i\mu_2)v - (\mu_3 + i\mu_4)(-\Delta)^\beta v + iu]dt + \beta_2 u dW_2(t), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.4}$$

with the same initial conditions and periodic boundary conditions to (1.1), where $\alpha, \beta \in (0,1)$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\beta_1 > 0$, $\mu_1 > 1 + \frac{\beta_2^2}{2}$, $\mu_3 > 0$, $\beta_2 > 0$, γ_3, σ_1, σ_2, μ_2 and μ_4 are real numbers and $W_1(t)$ and $W_2(t)$ are two-sided Wiener processes on a complete probability space.

The rest of the paper is organized as follows. The working function space and some basic concepts related to the random dynamical system are introduced in Section 2. We establish the well-posedness of the weak solutions for the deterministic CFGL equation in Section 3. In Section 4, we consider the long-time behavior of the solution and the existence of a global attractor is shown. Finally, a continuous random dynamical system for the stochastic CFGL equation is constructed and the existence of a random attractor is proved in Section 5.

2. Notations and preliminaries

In this section, we first review some notations for the working function space.

Denote

$$H = L^2_{\text{per}}(\mathcal{D}) = \{u | u \in L^2[0,2\pi], u(x+2\pi,t) = u(x,t)\}, \quad \mathcal{D} = [0,2\pi],$$
$$W = H \times H = \{(u,v) | u \in H, v \in H\},$$

with the norm

$$\|u\|_H^2 = \langle u, u^* \rangle = \int_{\mathcal{D}} |u|^2 dx, \qquad \|\phi\|_W^2 = \|u\|_H^2 + \|v\|_H^2,$$

where $\phi = (u,v) \in W$. For simplicity, we use the notation $\|\cdot\|$ to represent the norm for space H.

In what follows, we redefine some notations to the fractional derivative and fractional Sobolev space.

Since u is a periodic function, it can be expressed by a Fourier series

$$u(x) = (\mathcal{F}^{-1}\widehat{u})(x) := \frac{1}{2\pi}\sum_{\xi \in \mathbb{Z}} \widehat{u}(\xi) e^{i\xi x},$$

where

$$\widehat{u}(\xi) := \int_{\mathcal{D}} e^{-i\xi y} u(y) dy.$$

Then for $s \in \mathbb{R}$, denote

$$\Lambda^s u = \mathcal{F}^{-1}(|\xi|^s \widehat{u}(\xi)).$$

Finally, for any $s \in \mathbb{R}$, we define the homogeneous Sobolev space $\dot{H}^s$ under the norm

$$\|u\|_{\dot{H}^s} = \|\Lambda^s u\| = \left(\sum_{\xi \in \mathbb{Z}} |\xi|^{2s} |\widehat{u}(\xi)|^2 \right)^{\frac{1}{2}}.$$