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Critical polyharmonic problems with singular nonlinearities

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ABSTRACT

Let us consider the Dirichlet problem

$$\begin{cases} (-\Delta)^m u = \dfrac{|u|^{p_\alpha-2}u}{|x|^\alpha} + \lambda u & \text{in } \Omega \\ D^\beta u|_{\partial\Omega} = 0 & \text{for } |\beta| \le m-1 \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set containing the origin, $n > 2m$, $0 < \alpha < 2m$ and $p_\alpha = 2(n-\alpha)/(n-2m)$. We find that, when $n \ge 4m$, this problem has a solution for any $0 < \lambda < \Lambda_{m,1}$, where $\Lambda_{m,1}$ is the first Dirichlet eigenvalue of $(-\Delta)^m$ in Ω, while, when $2m < n < 4m$, the solution exists if λ is sufficiently close to $\Lambda_{m,1}$, and we show that these space dimensions are critical in the sense of Pucci–Serrin and Grunau. Moreover, we find corresponding existence and nonexistence results for the Navier problem, i.e. with boundary conditions $\Delta^j u|_{\partial\Omega} = 0$ for $0 \le j \le m-1$. To achieve our existence results it is crucial to study the behaviour of the radial positive solutions (whose analytic expression is not known) of the limit problem $(-\Delta)^m u = u^{p_\alpha-1}|x|^{-\alpha}$ in the whole space $\mathbb{R}^n$.

1. Introduction

In this paper we consider the following two critical growth polyharmonic problems: the first, with Dirichlet boundary conditions, is

$$\begin{cases} (-\Delta)^m u = \dfrac{|u|^{p_\alpha-2}u}{|x|^\alpha} + \lambda u & \text{in } \Omega \\ D^\beta u|_{\partial\Omega} = 0 & \text{for } |\beta| \le m-1; \end{cases} \tag{1.1}$$

the second, with Navier boundary conditions, is

$$\begin{cases} (-\Delta)^m u = \dfrac{|u|^{p_\alpha-2}u}{|x|^\alpha} + \lambda u & \text{in } \Omega \\ \Delta^j u|_{\partial\Omega} = 0 & \text{for } 0 \le j \le m-1. \end{cases} \tag{1.2}$$

Here $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ $(n > 2m)$ is a smooth bounded domain, $0 \in \Omega$, $\lambda \in \mathbb{R}$, $0 < \alpha < 2m$, and $p_\alpha = \frac{2(n-\alpha)}{n-2m}$. Moreover, in what follows we shall denote by $\Lambda_{m,1}$ the first Dirichlet eigenvalue of $(-\Delta)^m$ in Ω; let us recall that $\Lambda^m_{1,1}$ turns out to be the first Navier eigenvalue of $(-\Delta)^m$ in Ω.

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We prove the following results:

Theorem 1 (*Dirichlet Problem*)**.** *Let $n > 2m$, $\Omega \subset \mathbb{R}^n$ a bounded domain. Then:*

(i) *if $n \geq 4m$, the Dirichlet problem (1.1) has a non trivial weak solution $u \in H_0^m(\Omega)$ for any $\lambda \in (0, \Lambda_{m,1})$;*
(ii) *if $2m+1 \leq n \leq 4m-1$, there exists $\Lambda^* \in (0, \Lambda_{m,1})$ such that for any $\lambda \in (\Lambda^*, \Lambda_{m,1})$ the Dirichlet problem (1.1) has a non-trivial weak solution $u \in H_0^m(\Omega)$.*
(iii) *if $2m+1 \leq n \leq 4m-1$ and $\Omega = B =$ unit ball in $\mathbb{R}^n$, there exists $\Lambda_* \in (0, \Lambda_{m,1})$ such that problem (1.1) admits no positive weak solutions whenever $\lambda \in (0, \Lambda_*]$.*

Theorem 2 (*Navier Problem*)**.** *Let $n > 2m$, $\Omega \subset \mathbb{R}^n$ a bounded C^m-smooth domain. Then:*

(i) *if $n \geq 4m$, the Navier problem (1.2) has a non trivial positive weak solution $u \in H_\vartheta^m(\Omega)$ for any $\lambda \in (0, \Lambda_{1,1}^m)$;*
(ii) *if $2m+1 \leq n \leq 4m-1$, there exists $\Lambda^* \in (0, \Lambda_{1,1}^m)$ such that for any $\lambda \in (\Lambda^*, \Lambda_{1,1}^m)$ the Navier problem (1.2) has a non-trivial positive weak solution $u \in H_\vartheta^m(\Omega)$;*
(iii) *if $2m+1 \leq n \leq 4m-1$ and $\Omega = B =$ unit ball in $\mathbb{R}^n$, there exists $\widetilde{\Lambda}_* \in (0, \Lambda_{1,1}^m)$ such that problem (1.2) admits no positive radial weak solutions whenever $\lambda \in (0, \widetilde{\Lambda}_*]$.*

Only a few words of comment. In the non-singular case, i.e. when $\alpha = 0$ and $p_\alpha = 2^* = 2n/(n-2m)$, problems (1.1) and (1.2) have been widely studied, and our results are well known: for the Dirichlet problem see the starting point [1] in the case $m = 1$, [2] for $m = 2$ and finally [3,4] for any m; for the Navier problem see [5] for $m = 2$ and, for the general case, [6], which moreover contains a complete overview on the whole subject.

The main point in the present work is the lack of knowledge of the explicit analytical expression for the minimizers of the ratio

$$\frac{\int_{\mathbb{R}^n} |D^m u|^2 \, dx}{\left(\int_{\mathbb{R}^n} \frac{u^{p_\alpha}}{|x|^\alpha} \, dx\right)^{2/p_\alpha}}; \tag{1.3}$$

hence we need an accurate qualitative description of such minimizers to achieve the existence of nontrivial solutions to (1.1) and (1.2), following the scheme introduced by Brezis and Nirenberg in [1]. In general, when the analytical expression of a certain function U, minimizer of a suitable scale-invariant functional, is not available, careful estimates about the behaviour of U and some of its derivatives are needed to overcome this difficulty and gain existence theorems of nontrivial solutions to nonlinear critical problems (see for instance [7–9]).

Just another remark: statements (i), (ii) and (iii) in our theorems do not depend on α. In particular, as for statements (iii), the critical dimensions for our problems (in the sense of Pucci–Serrin [10] and Grunau [11]) are exactly $2m+1 \leq n \leq 4m-1$, i.e. the critical dimensions for the polyharmonic critical problem (both Dirichlet and Navier) when $\alpha = 0$ (see [11,6]). This perfectly fits with the theory that connects critical behaviour to L^2-summability of the fundamental solution of the linear operator (see [12,13]).

The paper is organized as follows: in Section 2 we study the linear problem $(-\Delta)^m u = f$ on $\mathbb{R}^n$ in the spherical symmetric case, and we provide a representation formula for the finite energy solutions (see Proposition 2.4) which is useful to study the nonlinear problem.

In Section 3 we study the behaviour of the minimizers of the ratio (1.3) which, up to multiplicative constants, are radial solutions to the so-called limit problem

$$(-\Delta)^m u = \frac{u^{p_\alpha - 1}}{|x|^\alpha}, \quad u > 0 \tag{1.4}$$

on the whole $\mathbb{R}^n$. The main results are contained in Theorem 3. Some (but not all) of the statements of Theorem 3 are known, and may be found in [14,15]. Nevertheless, we prefer to give a unitary (and more elementary) treatment of the subject; moreover, our approach allows us to state the behaviour of the derivatives of the minimizers of (1.3), which is essential in the sequel.

Once we have precise information about the minimizers of (1.3), the proof of our theorems is standard (see [6, Sections 7.5 and 7.6]). Hence, in Section 4 we sketch the proof of Theorem 1; the proof of Theorem 2 is quite similar, and we simply point out the small differences between the proofs.

Notations

- u^* The Schwarz symmetrization of u (see, for instance, [16]).
- 2^* $\frac{2n}{n-2m}$, the limit exponent for the Sobolev embedding $H^m(\Omega) \subset L^p(\Omega)$;
- p_α $\frac{2(n-\alpha)}{n-2m}$, $0 < \alpha < 2m$.