Nonlinear Analysis 110 (2014) 113–129

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Excess action and broken characteristics for Hamilton–Jacobi equations

Thomas Strömberg [a,*], Farzaneh Ahmadzadeh [b]

[a] Department of Engineering Sciences and Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden

[b] School of Innovation, Design and Engineering, Mälardalen University, SE-631 05 Eskilstuna, Sweden

ARTICLE INFO

Article history:
Received 20 March 2014
Accepted 1 August 2014
Communicated by S. Carl

MSC:
35F21
35A20
35D40
49L25

Keywords:
Hamilton–Jacobi equation
Generalized characteristic
Propagation of singularities

ABSTRACT

We study propagation of singularities for Hamilton–Jacobi equations

$$S_t + H(t, x, \nabla S) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

by means of the excess Lagrangian action and a related class of characteristics. In a sense, the excess action gauges how far a curve $\boldsymbol{X}(t)$ is from being action minimizing for a given viscosity solution $S(t, x)$ of the Hamilton–Jacobi equation. Broken characteristics are defined as curves along which the excess action grows at the slowest pace possible. In particular, we demonstrate that broken characteristics carry the singularities of the viscosity solution.

1. Introduction

The topic of this paper is the classical Hamilton–Jacobi equation

$$S_t + H(t, x, \nabla S) = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n, \tag{1}$$

$$\lim_{t \downarrow 0} S(t, x) = S_0(x) \quad \text{in } \mathbb{R}^n. \tag{2}$$

This nonlinear partial differential equation arises in, e.g., the calculus of variations, optimal control theory, and classical mechanics [1–4]. Regardless of the regularity of the data, the Cauchy problem (1)–(2) only rarely admits a classical solution. As is agreed upon, the appropriate concept of weak solution of (1) is that of a viscosity solution [5,4]. This notion of a generalized solution distinguishes the value function of an associated variational problem from other candidates of solutions. One of the overall goals of current research about (1)–(2) is to understand the time evolution of the singularities of a given viscosity solution of (1) defined as its points of nondifferentiability. In the one-dimensional case, for an equivalent scalar conservation law, Dafermos performed in [6] a detailed analysis showing among other things that singularities, once they form, propagate indefinitely along generalized characteristics (shock curves) forward in time. In the multidimensional case, Albano and Cannarsa defined in [7] a notion of generalized characteristics with respect to which they were able to establish similar results about the dynamics of singularities for (1). We recall their definition of generalized characteristics and main result.

* Corresponding author. Tel.: +49 929 491944; fax: +46 920 491073.
E-mail addresses: strom@ltu.se, thomas.stromberg@ltu.se (T. Strömberg), farzaneh.ahmadzadeh@mdh.se (F. Ahmadzadeh).

http://dx.doi.org/10.1016/j.na.2014.08.001
0362-546X/© 2014 Elsevier Ltd.

Definition 1. Let S be a viscosity solution of (1). A *generalized characteristic* in an interval $I \subseteq [0, \infty)$ is a locally Lipschitz continuous curve $\boldsymbol{X}: I \to \mathbb{R}^n$ such that the differential inclusion

$$\dot{\boldsymbol{X}}(t) \in \operatorname{co} \nabla_p H(t, \boldsymbol{X}(t), \nabla^+ S(t, \boldsymbol{X}(t))) \tag{3}$$

holds for almost all $t \in I$.

The *singular set* of S is defined as the set Σ of points where S fails to be differentiable.

Theorem 1. *Suppose that $H \in C^1$ and $p \mapsto H(t, x, p)$ is strictly convex for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$. Let S be a locally semiconcave viscosity solution of (1). Assume that $(t_0, x_0) \in \Sigma$. Then there exist $t_1 > t_0$ and a generalized characteristic $\boldsymbol{X}$ in the time interval $[t_0, t_1)$ such that $\boldsymbol{X}(t_0) = x_0$ and $(t, \boldsymbol{X}(t)) \in \Sigma$ for all $t \in [t_0, t_1)$.*

Proof. See Theorem 8 in [7], Theorems 5.6.6 in [2], or [8]. A refined result appears in Theorem 3.2 of [9]. □

The definition of generalized characteristics is weak as the differential inclusion (3) actually involves two convexification operations: an outer convexification as well as an inner one appearing implicitly through the superdifferential $\nabla^+ S$. Accordingly, when $n \geq 2$, there may exist several generalized characteristics emanating from a given singular point [9,10]. The purpose of this paper is to study a much stronger notion of characteristics entailing a more accurate description of the spreading of singularities for Hamilton–Jacobi equations. In their articles [11,12], Khanin and Sobolevski made substantial progress on the problem of singular dynamics by proving the existence of what we have chosen to call "broken characteristics". They were motivated by a fluid dynamical analogy and proved the existence of a broken characteristic emanating from a given point. Broken characteristics were viewed as particle trajectories which may hit $\overline{\Sigma}$ and after that will move inside $\overline{\Sigma}$. The dynamics in the singular set Σ is governed by a specific law. Before intersecting the singular support $\overline{\Sigma}$, trajectories are unique minimizers of the classical Lagrangian action and simultaneously classical characteristics as well as extremals. After hitting $\overline{\Sigma}$, while ceasing to minimize the Lagrangian action, they will continue their motion inside the singular set, or in its closure, in such a way that the "excess action" grows as slowly as possible.

We summarize the contribution of Khanin and Sobolevski. As a first step, they defined at each $(t, x) \in Q$ an "admissible momentum" $p^*(t, x)$ and an "admissible velocity" $v^*(t, x)$ related by $p^*(t, x) = \nabla_v L(t, x, v^*(t, x))$. The admissible velocity arises as the minimum point of a certain strictly convex function, $v \mapsto Y(t, x, v)$, which is derived from the Lagrangian and the superdifferential of the value function S. At any nonsingular point (t, x), the admissible momentum and velocity are given by the classical expressions $p^*(t, x) = \nabla S(t, x)$ and $v^*(t, x) = \nabla_p H(t, x, \nabla S(t, x))$. In general, p^* is a particular selection of the multivalued superdifferential $\nabla^+ S$. Second, Khanin and Sobolevski established the existence of a nonsmooth flow that is compatible with the discontinuous velocity field v^*. Namely, they demonstrated that for every point $(t_0, x_0) \in Q$ there exists a locally Lipschitz curve $\boldsymbol{X}: [t_0, t_1) \to \mathbb{R}^n$ with $\boldsymbol{X}(t_0) = x_0$ whose right derivative $\dot{\boldsymbol{X}}_+(t)$ exists and satisfies $\dot{\boldsymbol{X}}_+(t) = v^*(t, \boldsymbol{X}(t))$ for every $t \in [t_0, t_1)$. Such trajectories are called "broken characteristics" in this paper. They carry the singularities of S (as verified in Theorem 2). Broken characteristics are, needless to say, very special generalized characteristics.

This paper investigates in some detail the relation between broken characteristics and excess action. We show that the time derivative of the excess action along a given curve can be described in terms of the nonnegative function Y. In this picture, broken characteristics are trajectories along which the excess action grows at the slowest pace possible. The propagation property stated in Theorem 1 remains fulfilled for broken characteristics (see Theorem 2). We also obtain results about the right-continuity of $\dot{\boldsymbol{x}}_+$ and other functions (see Theorem 2, Corollary 1) and refine the description of the singularity propagation (Theorems 4 and 5).

2. Conditions and prerequisites

The Hamiltonian H appearing in (1) is the Legendre–Fenchel transform of a Lagrangian L.

2.1. Conditions

We assume the following conditions connecting (1) to a well-behaved problem in the calculus of variations.

(A) The Lagrangian $L(t, x, v)$ is $C^2([0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n)$. It is coercive in the sense that $L(t, x, v) \geq \ell(|v|)$ for all $(t, x, v) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ where $\ell(s)/s \to \infty$ as $s \to \infty$. Furthermore, $L(t, x, v)$ is a locally uniformly convex function of $v \in \mathbb{R}^n$ for every $(t, x) \in [0, \infty) \times \mathbb{R}^n$, i.e., the Hessian matrix $\nabla_v^2 L(t, x, v)$ is positive definite for all $(t, x, v) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$. The Hamiltonian H is given by

$$H(t, x, p) = \max_{v \in \mathbb{R}^n} (\langle p, v \rangle - L(t, x, v)), \quad (t, x, p) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n.$$

(B) At least one of conditions (i) and (ii) below is satisfied:
   (i) For any $r > 0$ and $T > 0$ there exist a nonnegative constant k and a nonnegative $\gamma \in L^1(0, T)$ such that
   $$|L(t, x, v) - L(\tilde{t}, x, v)| \leq (k|L(t, x, v)| + \gamma(t))|t - \tilde{t}|$$
   for all $t, \tilde{t} \in [0, T]$, $|x| \leq r$ and $v \in \mathbb{R}^n$.