Nonlinear Analysis 110 (2014) 170–181

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Liapunov-type inequalities for third-order half-linear equations and applications to boundary value problems

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ARTICLE INFO

Article history:
Received 2 May 2014
Accepted 29 July 2014
Communicated by Enzo Mitidieri

Keywords:
Liapunov-type inequality
Third-order
Half-linear differential equations
Boundary value problems
Existence and uniqueness of solutions

ABSTRACT

We derive Liapunov-type inequalities for third-order half-linear differential equations. These inequalities utilize integrals of both $q_+(t)$ and $q_-(t)$ rather than those of $|q(t)|$ as in most papers in the literature for higher-order Liapunov-type inequalities. Furthermore, by combining these inequalities with the "uniqueness implies existence" theorems by several authors, we establish the uniqueness and hence existence–uniqueness for several classes of boundary value problems for third-order linear equations. We believe that this is the first time for Liapunov-type inequalities to be used to deal with boundary value problems and expect that this approach can be further applied to study general higher-order boundary value problems.

1. Introduction

For the second-order linear differential equation

$$x'' + q(t)x = 0, \tag{1.1}$$

where $q \in C([a, b], \mathbb{R})$, the following result is known as the Liapunov inequality, see [1,2].

Theorem 1.1. *Assume $x(t)$ is a solution of Eq. (1.1) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$\int_a^b |q(t)|\, dt > \frac{4}{b-a}. \tag{1.2}$$

It was first noticed by Wintner [3] and later by several other authors that inequality (1.2) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{q(t), 0\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}. \tag{1.3}$$

The Liapunov inequality was extended by Hartman [4, Chapter XI] to the more general equation

$$\left(r(t)x'\right)' + q(t)x = 0, \tag{1.4}$$

where $q, r \in C([a, b], \mathbb{R})$ such that $r(t) > 0$ for $t \in [a, b]$, as follows:

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http://dx.doi.org/10.1016/j.na.2014.07.020
0362-546X/

Theorem 1.2. *Assume $x(t)$ is a solution of Eq. (1.4) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$\int_a^b q_+(t)\,dt > \frac{4}{\int_a^b r^{-1}(t)\,dt}. \tag{1.5}$$

The above inequalities have been further improved by replacing $\int_a^b q_+(t)\,dt$ by some integrals of $q(t)$ on parts of or the whole interval $[a, b]$. In fact, under the assumptions of Theorem 1.1, Harris and Kong [5, Theorem 2.3] showed that there exist two disjoint subintervals I_1 and I_2 such that

$$\int_{I_1\cup I_2} q(t)\,dt > \frac{4}{b-a} \quad \text{and} \quad \int_{[a,b]\setminus(I_1\cup I_2)} q(t)\,dt \le 0,$$

and Brown and Hinton [6] showed that

$$\left|\int_a^b q(t)\,dt\right| > \frac{4}{b-a}.$$

We note that the number 4 in the above inequalities is the best in the sense that if it is replaced by any larger number, then the inequalities fail to hold, see [4, p. 345] and [7] for examples.

In recent years, Liapunov-type inequalities have also been developed for higher-order differential equations. In particular, Parhi and Panigrahi [8] established the Liapunov-type inequalities for the third-order linear differential equation

$$x''' + q(t)x = 0 \tag{1.6}$$

as follows:

Theorem 1.3. (a) *Let $-\infty < a < b < \infty$ and $q \in C([a, b], \mathbb{R})$. Assume $x(t)$ is a solution of Eq. (1.6) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. If there exists a $\xi \in [a, b]$ such that $x''(\xi) = 0$, then*

$$\int_a^b |q(t)|\,dt > \frac{4}{(b-a)^2}.$$

(b) *Let $-\infty < a < b < c < \infty$ and $q \in C([a, c], \mathbb{R})$. Assume $x(t)$ is a solution of Eq. (1.6) with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for all $t \in (a, b) \cup (b, c)$. Then*

$$\int_a^c |q(t)|\,dt > \frac{4}{(c-a)^2}.$$

This result was further extended to general nth order linear equations by Yang [9], Cekmak [10], and Yang and Lo [11], and Zhang and He [12]. For more Liapunov-type inequalities, we refer the reader to Guseinov and Kaymakcalan [13] and Guseinov and Zafer [14] for linear Hamiltonian systems; to Pachpatte [15], Tiryaki, Unal and Cekmak [16,17], and Cekmak [18] for nonlinear equations and systems; and to Jiang and Zhou [19] and Unal and Cekmak [20] for equations and systems on time scales.

An extension of the Liapunov inequality to the second-order half-linear differential equations was done by Elbert [21] and Dosly and Rehak [22]. Consider the equation

$$\left(r(t)\,\phi_\lambda\left(x'\right)\right)' + q(t)\,\phi_\lambda(x) = 0, \tag{1.7}$$

where $q, r \in C([a, b], \mathbb{R})$ such that $r(t) > 0$ for $t \in [a, b]$, and $\phi_\lambda(x) = |x|^{\lambda-1}x$ with $\lambda > 0$. We state their result in the next theorem.

Theorem 1.4. *Assume $x(t)$ is a solution of Eq. (1.7) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$\int_a^b q_+(t)\,dt > \frac{2^{\lambda+1}}{\left(\int_a^b r^{-\frac{1}{\lambda}}(t)\,dt\right)^{\lambda}}. \tag{1.8}$$

However, Liapunov-type inequalities have not been well-developed for higher-order half-linear differential equations. This is due to the complexity caused by the nonlinear nature of the Laplacian operators in the equation.

In this paper, we will establish Liapunov-type inequalities for third-order half-linear equations. We point out that our inequalities utilize integrals of both $q_+(t)$ and $q_-(t)$, the positive and negative parts of $q(t)$; which is different from most existing results for higher-order cases, where integrals of $|q(t)|$ are involved. As a special case, our work improves the result in Theorem 1.3 for the linear case.