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Global gradient estimates for non-quadratic vector-valued parabolic quasi-minimizers

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ABSTRACT

We establish local and global higher integrability estimates for the gradient of vector-valued parabolic quasi-minimizers of integral functionals with polynomial p-growth, $p > \frac{2n}{n+2}$, on a parabolic domain Ω_T in $\mathbb{R}^{n+1}$, $n \geq 2$ and under a weak regularity assumption on the parabolic boundary of Ω_T.

1. Introduction and statement of the results

1.1. The problem and assumptions

On a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and a parabolic cylinder $\Omega_T := \Omega \times (0, T)$ with $T > 0$ we consider so-called **parabolic quasi-minimizers**. In this context let $f \equiv f(z, u, w) : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a Carathéodory function which fulfills a general growth condition of the type

$$\nu|\zeta|^p - L_1 \leq f(z, \xi, \zeta) \leq L|\zeta|^p + L_1, \quad \text{with } p > \frac{2n}{n+2}, \tag{1.1}$$

for all $z \in \Omega_T$, $\xi \in \mathbb{R}^N$ and $\zeta \in \mathbb{R}^{Nn}$, and with constants $0 < \nu \leq 1 \leq L < \infty$ and $L_1 \geq 0$. We are mainly interested in vector-valued functions, $N > 1$, but the results hold also for the case $N = 1$ and for large parts of the following discussions we will merely assume the above stated growth assumption for the integrand f, in particular no further regularity for example on the dependence upon the space–time variables (x, t) and no differentiability with respect to the gradient variable are assumed. A model case for our framework is a parabolic quasiminimizer related to the Dirichlet p-energy

$$\int_{\Omega_T} |Du|^p \, dz,$$

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for functions $u: \Omega_T \to \mathbb{R}^N$, which correspond in the case of real minimizers (quasi-minimizers with constant $\mathcal{Q} = 1$, see Definition 1.1) to solutions of the parabolic p-Laplace system

$$\partial_t u - \operatorname{div}\big(|Du|^{p-2}Du\big) = 0 \quad \text{on } \Omega_T,$$

under appropriate initial–boundary conditions. However, assuming merely the growth condition (1.1), we act in a much more general framework. In particular, since we are considering $\mathcal{Q}$-minimizers with $\mathcal{Q} > 1$, there is no corresponding system of PDEs and we therefore have to focus on purely variational techniques for proving regularity issues.

1.1.1. Local and global quasi-minimizers

We distinguish between local and global quasi-minimizers. In the case of local quasi-minimizers we do not have to take care of the situation on the parabolic boundary of the cylinder Ω_T, since we consider only compactly contained subsets of Ω_T and we define a local quasi-minimizer as follows:

Definition 1.1. For given fixed $\mathcal{Q} \geq 1$ we call the function $u \in L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N))$ a *(local) parabolic quasi-minimizer or $\mathcal{Q}$-minimizer* of the integral functional

$$\mathcal{F}[w, \Omega_T] := \int_{\Omega_T} f(z, w, Dw)\, dz, \tag{1.2}$$

if for all test functions $\Phi \in C^\infty_c(\Omega_T; \mathbb{R}^N)$ the inequality

$$-\int_K \langle u, \partial_t \Phi\rangle \, dz + \mathcal{F}[u, K] \leq \mathcal{Q} \cdot \mathcal{F}[u - \Phi, K], \tag{1.3}$$

holds true, with $K = \operatorname{spt} \Phi$. Here $\langle \cdot, \cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^N$, $\partial_t\varphi \equiv \varphi_t \equiv \frac{\partial\varphi}{\partial t}$ denotes the time derivative and $Du = \nabla u$ the derivative with respect to the space variables in the distributional sense. For a fixed subset $B \subset C^\infty_c(\Omega_T)$ we call u a *restricted $\mathcal{Q}$-minimizer on B*, if the inequality (1.3) is satisfied only for all test functions $\Phi \in B$.

In this context we define the parabolic Sobolev space $L^p(0, T; W^{1,q}(\Omega; \mathbb{R}^N))$ for parameters $1 < p, q < \infty$ as the space of all measurable functions $u : \Omega_T \to \mathbb{R}^N$ which fulfill $u(\cdot, t) \in W^{1,q}(\Omega; \mathbb{R}^N)$ for almost all $t \in (0, T)$ and for which the scalar function $t \mapsto \|u(\cdot, t)\|_{W^{1,q}(\Omega)}$ is in $L^p(0, T)$. In a standard way – by considering compactly contained subsets $K \Subset \Omega_T$ – we can define the local variants $L^p_{\text{loc}}(0, T; W^{1,q}_{\text{loc}}(\Omega; \mathbb{R}^N))$ of these spaces. By an approximation argument, we can use in the defining inequality (1.3) test functions

$$\Phi \in L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega)) \quad \text{with } \partial_t \Phi \in L^{p'}_{\text{loc}}(\Omega_T),$$

where $p' = p/(p-1)$ denotes the Hölder conjugate of p.

If we fix a function $g \in W^{1,p}(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ on the cylinder Ω_T, we can also consider global quasi-minimizers which satisfy in a certain sense the initial–boundary-condition '$u = g$' on the parabolic boundary $\partial_{\text{par}}\Omega_T := \Omega \times \{0\} \cup \partial\Omega \times (0, T)$. Here, the space $W^{1,p}(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ is defined as

$$W^{1,p}(0, T; W^{1,p}(\Omega; \mathbb{R}^N)) := \big\{u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N)) \colon \partial_t u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N))\big\},$$

where $\partial_t u$ denotes the weak time derivative of u, in the sense that

$$-\int_0^T u\,\varphi'\, dt = \int_0^T u'\varphi\, dt \quad \text{for all } \varphi \in C^\infty_o((0, T)).$$

$C^\infty_o((0, T))$ denotes the space of all scalar valued test functions φ with support $\operatorname{spt}\varphi \subseteq (0, T)$. A global parabolic $\mathcal{Q}$-minimizer is then defined as follows:

Definition 1.2. For given fixed $\mathcal{Q} \geq 1$ a function $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ is called a *global parabolic $\mathcal{Q}$-minimizer*, if the inequality (1.3) holds for all test functions $\Phi \in C^\infty_c(\Omega_T; \mathbb{R}^N)$ and if moreover the initial- and boundary-conditions

$$u(\cdot, t) - g(\cdot, t) \in W^{1,p}_o(\Omega; \mathbb{R}^N), \quad \text{for almost all } t \in (0, T), \tag{1.4}$$

and

$$\frac{1}{h}\int_0^h \int_\Omega |u - g|^2 \, dx\, dt \to 0, \quad \text{as } h \downarrow 0 \tag{1.5}$$

hold true.

Remark 1.3. Since $g \in W^{1,p}(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$, the mapping $g\colon [0, T] \to W^{1,p}(\Omega; \mathbb{R}^N)$ has an absolutely continuous representative (see for example [34, p. 193ff] for a detailed discussion) We choose this representative to give sense to the above definition.

Remark 1.4. Again, by a standard approximation scheme, in view of the growth condition (1.1), we can use as test function in the defining inequality (1.2) any function $\Phi \in L^p(0, T, W^{1,p}_0(\Omega))$ with $\partial_t \Phi \in L^{p'}(\Omega_T)$.