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Positive solutions for some indefinite nonlinear eigenvalue elliptic problems with Robin boundary conditions

Humberto Ramos Quoirin[a], Antonio Suárez[b,*]

[a] Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

[b] Universidad de Sevilla, Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Calle Tarfia s/n, 41012-Sevilla, Spain

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ABSTRACT

We consider a nonlinear eigenvalue problem with indefinite weight under Robin boundary conditions. We prove the existence and multiplicity of positive solutions. To this end, we carry out a detailed study of some linear eigenvalues problems and we use mainly bifurcation and sub–supersolution methods.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with a $C^{2,\gamma}$ boundary, $0 < \gamma < 1$. We are interested in the existence and stability properties of positive solutions for the problem

$$\begin{cases} -\Delta u = \lambda m(x)(u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\lambda, \alpha \in \mathbb{R}$, $m \in C^1(\overline{\Omega})$ changes sign and ν is the outward unit normal to $\partial\Omega$.

Throughout this article we assume that

$$\int_\Omega m < 0, \tag{1}$$

since the case $\int_\Omega m > 0$ reduces to (1) changing λ by $-\lambda$. The case $\int_\Omega m = 0$ is singular and will be treated elsewhere.

We shall treat (P) by a bifurcation approach, so we shall consider the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda m(x) u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega. \end{cases} \tag{E}$$

* Corresponding author.
E-mail addresses: humberto.ramos@usach.cl (H. Ramos Quoirin), suarez@us.es (A. Suárez).

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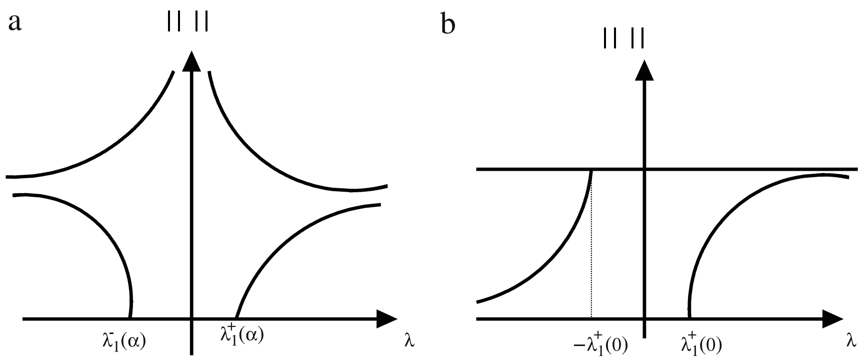

Fig. 1. Bifurcation diagrams of (P): Case (a) $\alpha < 0$ and Dirichlet boundary conditions. Case (b) $\alpha = 0$.

It is known by [1] that there exists $\alpha_0^* > 0$ such that (E) possesses two principal eigenvalues, denoted by $\lambda_1^-(\alpha)$ and $\lambda_1^+(\alpha)$, for $\alpha < \alpha_0^*$. In the homogeneous Dirichlet boundary conditions case, we denote them by $\lambda_1^\pm(D)$. In Section 2 we recall the results from [1] and complement them providing an expression for α_0^*.

Let us note that (P) has already been studied in different cases. For the cases $\alpha < 0$ (cf. [7]) and Dirichlet boundary conditions (cf. [2,11]), it has been proved that (P) has a positive solution for all $\lambda \neq 0$ and, under further conditions for a priori bounds, at least two positive solutions for $\lambda \in (-\infty, \lambda_1^-(\alpha)) \cup (\lambda_1^+(\alpha), +\infty)$ and $\lambda \in (-\infty, \lambda_1^-(D)) \cup (\lambda_1^+(D), +\infty)$, respectively. See Fig. 1(a) for the bifurcation diagram in these cases.

The case $\alpha = 0$, which has been analyzed in [6] (see also [14,17]), is singular in the following sense: the trivial solutions $u \equiv 0$ and $u \equiv 1$ exist for all $\lambda \in \mathbb{R}$, and for $\lambda = 0$ the positive constants are solutions. Moreover, for $\lambda \in (-\infty, -\lambda_1^+(0)) \cup (\lambda_1^+(0), +\infty)$ there exists a stable solution $u < 1$, which is the only positive solution of (P) less than one, see Fig. 1(b). Recall that in this case $\lambda_1^-(0) = 0$.

Finally, the case $\alpha > 0$ and small was studied in [7]. Assuming $2 < (N+2)/(N-2)$ and using variational methods, the authors proved that if $0 < \alpha < \alpha_0^*$ and $\lambda \in (\lambda_1^-(\alpha), \lambda_1^+(\alpha))$ then (P) possesses at least a positive solution.

In this article, we adopt a different viewpoint, namely, we fix λ and look at α as a bifurcation parameter. Consequently, we improve some results of [6] for $\alpha = 0$, and complement the study of (P) when $\alpha > 0$.

We shall assume that

$$M_\pm := \{x \in \Omega : m^\pm > 0\}$$

are open and regular sets; here $m^\pm = \max\{\pm m, 0\}$. We shall also assume that $m^\pm(x) \approx [\mathrm{dist}(x, \partial M_\pm)]^{\gamma^\pm}$ for x close to $\partial M_\pm$ and some $\gamma_\pm \geq 0$.

Let

$$M_0 := \Omega \setminus (\overline{M_+} \cup \overline{M_-}). \tag{2}$$

We assume the following conditions on $M_\pm$ and M_0:

$$M_0 \text{ is a proper subdomain of } \Omega, \quad \text{i.e. } dist(\partial\Omega, \partial M_0 \cap \Omega) > 0. \tag{$\mathrm{H_{M_0}}$}$$

$$\partial M_\pm = \Gamma_1^\pm \cup \Gamma_2^\pm, \quad \text{with } \Gamma_1^\pm = \partial\Omega \cap \partial M_\pm \text{ and } \Gamma_2^\pm \subset \Omega. \tag{$\mathrm{H_{M_\pm}}$}$$

In fact, ($\mathrm{H_{M_\pm}}$) is assumed to avoid regularity issues, see [12]. Two examples of domains satisfying the above conditions are depicted in Fig. 2.

Our first result is related to *a priori* bounds for positive solutions of (P). We show that if

$$2 < \min\left\{\frac{N+2}{N-2}, \frac{N+1+\gamma^\pm}{N-1}\right\}, \tag{3}$$

then, there exist *a priori* bounds for positive solutions of (P) whenever α varies in compact sets of $\mathbb{R}$.

In order to state our main results, we need to introduce some further notation. We denote by $\lambda_1(-\Delta - \lambda m, N)$ and $\lambda_1(-\Delta - \lambda m, D)$ the principal eigenvalues of the problem

$$-\Delta\varphi - \lambda m(x)\varphi = \sigma\varphi \quad \text{in } \Omega,$$

under homogeneous Neumann and Dirichlet boundary conditions, respectively. In Section 2, we show that given $\lambda \in \mathbb{R}$, there exists a principal eigenvalue of (E) with respect to α, denoted by $\alpha_1(\lambda)$, if and only if $\lambda_1(-\Delta - \lambda m, D) > 0$. Furthermore, $sign(\alpha_1(\lambda)) = sign(\lambda_1(-\Delta - \lambda m, N))$.

Note that if $\lambda = 0$ then (P) has no positive solutions unless $\alpha = 0$, in which case the positive constants are solutions. So we assume that $\lambda \neq 0$ throughout this article.

We state now our main results (see Fig. 3):